PHIL309P
Philosophy, Politics and Economics

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Taking stock

- Aggregating judgements: single event, multiple issues, logically connected issues, probabilistic opinions, imprecise probabilities, causal models, ...
- May's Theorem: axiomatic characterization of majority rule
- Condorcet Jury Theorem: epistemic analysis of majority rule
- Aggregation paradoxes: multiple election paradox, doctrinal paradox, discursive dilemma, the problem with conjunction, the corroboration paradox

Judgement Aggregation

U. Endriss. *Judgment Aggregation*. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, Handbook of Computational Social Choice, Cambridge University Press, 2016.

C. List. *The theory of judgment aggregation: An introductory review*. Synthese 187(1): 179-207, 2012.

D. Grossi and G. Pigozzi. *Judgement Aggregation: A Primer*. Morgan & Claypool Publishers, 2014.

Vote by Grading

Approval Voting: Each voter selects a subset of candidates. The candidate with the most "approvals" wins the election.

S. Brams and P. Fishburn. *Approval Voting*. Birkhauser, 1983.

J.-F. Laslier and M. R. Sanver (eds.). *Handbook of Approval Voting*. Studies in Social Choice and Welfare, 2010.

Under Approval Voting (AV), voters are asked to select the candidates that the voter *approves*.

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Approving of a candidate is not (necessarily) the same as simply ranking the candidate first.

Why Approval Voting?

www.electology.org/approval-voting

S. Brams and P. Fishburn. *Going from Theory to Practice: The Mixed Success of Approval Voting*. Handbook of Approval Voting, pgs. 19-37, 2010.

Approval Voting is more flexible

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$

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# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$, vote-for-2 elects $\{D\}$

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There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$, vote-for-2 elects $\{D\}$, vote-for-3 elects $\{A, B\}$.

Approval Voting is more flexible

AV may elect the Condorcet winner

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
$(\{A\}, \{B\}, \{C, A\})$ elects A under AV.

Possible Failure of Unanimity

# voters	1	1	1
	A	C	D
	B	A	A
	C	B	B
	D	D	C

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# voters	1	1	1
	A	C	D
	B	A	A
	C	B	B
	D	D	C

Approval Winners: A, B

Indeterminate or Responsive?

# voters	6	5	4
	A	B	C
	C	C	B
	B	A	A

Plurality winner: A, Borda and Condorcet winner: C.

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# voters	6	5	4
	A	B	C
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	B	A	A

Plurality winner: A, Borda and Condorcet winner: C.
Any combination of A, B and C can be an AV winner (or AV winners).

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Ask the voters to provide both a linear ranking of the candidates and the candidates that they approve.

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Make the ballots more expressive: Dis&Approval voting, RangeVoting, Majority Judgement

Grading

In many group decision situations, people use measures or grades from a **common language of evaluation** to evaluate candidates or alternatives:

- in figure skating, diving and gymnastics competitions;
- in piano, flute and orchestra competitions;
- in classifying wines at wine competitions;
- in ranking university students;
- in classifying hotels and restaurants, e.g., the Michelin $*$

Voting by Grading: Questions

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Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)
- How should we *aggregate* the grades? (e.g., Average or Median)
- Should there be a “no opinion” option?

Voting by Grading: Examples

Approval Voting: voters can assign a single grade “approve” to the candidates

Dis&Approval Voting: voters can approve or disapprove of the candidates

Majority Judgement, Score Voting: voters can assign any grade from a fixed set of grades to the candidates

Strong Paradox of Grading Systems

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	5/3
B	0	3	1	4/3
C	0	3	1	4/3

Average Grade Winner: A

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$B \succ A$$

Grades: $\{0, 1, 2, 3\}$

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3 Voters

# voters	1	1	1	Avg
A	3	2	0	
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C	0	3	1	

Average Grade Winner: A

$$C \sim B \succ A$$

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# voters	1	1	1	Avg
A	3	2	0	
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Average Grade Winner: A

$$C \sim B \succ A$$

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

Superior Grade Winners: C, B

Grades: $\{0, 1, 2, 3, 4, 5\}$

Candidates: $\{A, B, C\}$

5 Voters

# voters	1	4	Avg
A	5	0	5/5
B	0	1	4/5
C	0	1	4/5

Average Grade Winner: A

Superior Grade Winner: B, C

To conclude, we have identified a paradox of grading systems, which is not just a mirror of the well-known differences that crop up in aggregating votes under ranking systems. Unlike these systems, for which there is no accepted way of reconciling which candidate to choose when, for example, the Hare, Borda and Condorcet winners differ, AV provides a solution when the AG and SG winners differ.

Theorem (Brams and Potthoff). When there are two grades, the AG and SG winners are identical.

Re-examining the the social choice problem: Maximizing *social welfare*

Social Utility?

Utilitarianism (Bentham, Mill, etc.): Place at the top the social options that produce the *greatest amount of pleasure for the citizenry as a whole*

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How are we to *measure* the amount of pleasure available under each social option?

A reminder on modern utility theory...

Utility Function

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

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What properties does such a preference ordering have?

$$X = \{M, C, P, L\}$$

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$M\ C\ P\ L$	$M\ C$	$P\ L$
$M\ P\ L$	$M\ P$	M
$C\ P\ L$	$M\ L$	C
$M\ C\ P$	$C\ P$	P
$M\ C\ L$	$C\ L$	L

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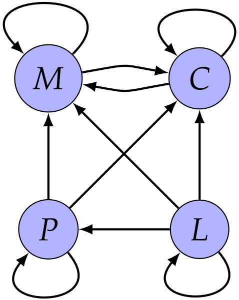
$\boxed{M\ C}\ P\ L$	$\boxed{M\ C}$	$\boxed{P}\ L$
$\boxed{M}\ P\ L$	$\boxed{M}\ P$	$\boxed{M}$
$\boxed{C}\ P\ L$	$\boxed{M}\ L$	$\boxed{C}$
$\boxed{M\ C}\ P$	$\boxed{C}\ P$	$\boxed{P}$
$\boxed{M\ C}\ L$	$\boxed{C}\ L$	$\boxed{L}$

$$X = \{M, C, P, L\}$$



$$\succeq = \{(M, C), (C, M), (M, P), (M, L), (C, P), (C, L), (P, L),$$
$$(M, M), (P, P), (C, C), (L, L)\}$$

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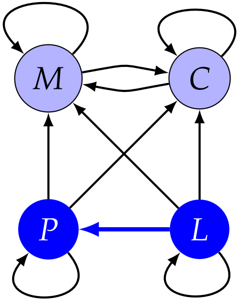

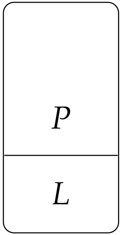

$$\succeq = \{(M,C),(C,M),(M,P),(M,L),(C,P),(C,L),(P,L),$$
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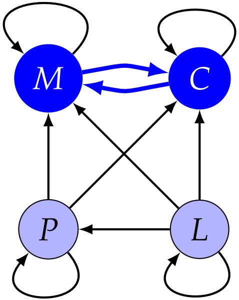

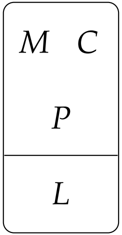

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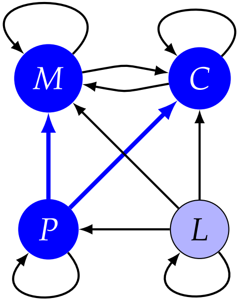

M	C
P	
L	

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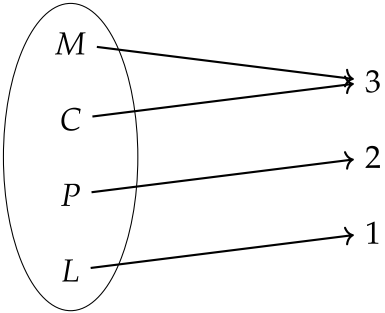

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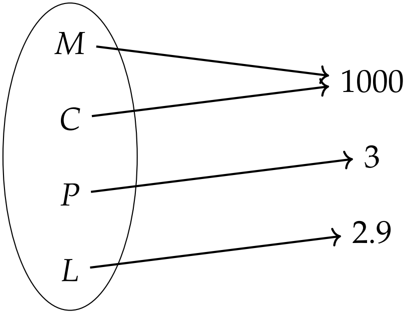

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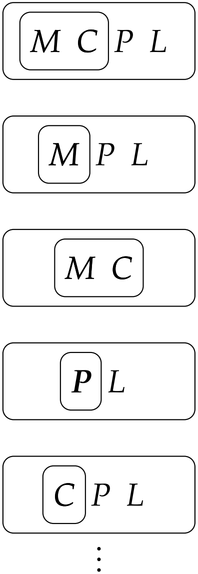

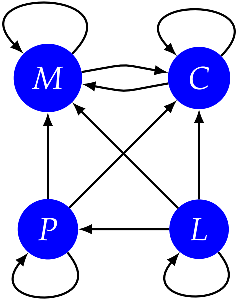

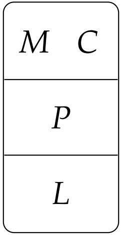

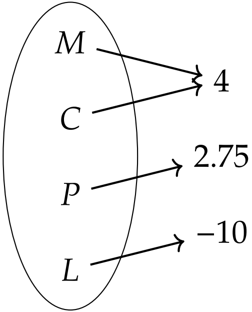

Important

All three of the utility functions represent the preference $x \succ y \succ z$

Item	u_1	u_2	u_3
x	3	10	1000
y	2	5	99
z	1	0	1

$x \succ y \succ z$ is represented by both $(3, 2, 1)$ and $(1000, 999, 1)$, so one cannot say that y is “closer” to x than to z.

Ordinal vs. Cardinal Utility

Ordinal scale: Qualitative comparisons of objects allowed, no information about differences or ratios.

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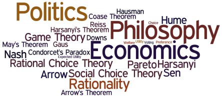

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Cardinal scales:

> **Interval scale**: Quantitative comparisons of objects, accurately reflects differences between objects.
>
> E.g., the difference between 75°F and 70°F is the same as the difference between 30°F and 25°F However, 70°F (= 21.11°C) is **not** twice as hot as 35°F (= 1.67°C). The difference between 70°F and 65°F is **not** the same as the difference between 25°C and 20°C.

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>
> **Ratio scale**: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10lb is twice as much as 5lb. But, 10kg is not twice as much as 5lb.

Suppose that X is a set of outcomes.

A **(simple) lottery** over X is denoted $[x_1 : p_1, x_2 : p_2, \ldots, x_n : p_n]$ where for $i = 1, \ldots, n$, $x_i \in X$ and $p_i \in [0, 1]$, and $\sum_i p_i = 1$.

Let $\mathcal{L}$ be the set of (simple) lotteries over X. We identify elements $x \in X$ with the lottery $[x : 1]$.

Suppose that $\succeq$ is a relation on $\mathcal{L}$.

Axioms

Preference $\succeq$ is reflexive, transitive and complete

Compound Lotteries The decision maker is indifferent between every compound lottery and the *corresponding* simple lottery.

Independence For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0, 1]$, $L_1 \succ L_2$
if, and only if,
$[L_1 : a, L_3 : (1 - a)] \succ [L_2 : a, L_3 : (1 - a)]$.

Continuity For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0, 1]$,
if $L_1 \succ L_2 \succ L_3$, then there exists $a \in (0, 1)$
such that $[L_1 : a, L_3 : (1 - a)] \sim L_2$

$u : \mathcal{L} \to \mathfrak{R}$ is linear provided for all $L = [L_1 : p_1, \ldots, L_n : p_n] \in \mathcal{L}$,

$$u(L) = \sum_{i=1}^{n} p_i u(L_i)$$

von Neumann-Morgenstern Representation Theorem A binary relation $\succeq$ on $\mathcal{L}$ satisfies Preference, Compound Lotteries, Independence and Continuity iff $\succeq$ is representable by a linear utility function $u : \mathcal{L} \to \mathfrak{R}$.

Moreover, $u' : \mathcal{L} \to \mathfrak{R}$ represents $\succeq$ iff there exists real numbers $c > 0$ and d such that $u'(\cdot) = cu(\cdot) + d$. ("$u$ is unique up to linear transformations.")

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- Issue with continuity: 1EUR $\succ$ 1 cent $\succ$ death, but who would accept a lottery which is p for 1EUR and $(1 - p)$ for death??

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- Utility is unique only *up to linear transformations*. So, it still does not make sense to add two different agents cardinal utility functions.
- Issue with continuity: 1EUR $\succ$ 1 cent $\succ$ death, but who would accept a lottery which is p for 1EUR and $(1 - p)$ for death??
- Important issues about how to identify correct descriptions of the outcomes and options.

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Measures of Social Utility:

- Sum Utilitarian: maximize $\sum_i u_i$
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- Egalitarian: maximize $\min_i\{u_i\}$
- Nash: maximize $\Pi_i u_i$

	v_1	v_2	$U(\cdot)$
A	0	1	1
B	1	–1	0
C	–1	0	–1

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B	1	–1	0
C	–1	0	–1

	v_1	v_2	$U(\cdot)$
A	0	1	1
B	10	–1	9
C	–10	0	–10

	v_1	v_2	$U(\cdot)$
A	0	1	1
B	1	–1	0
C	–1	0	–1

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	v_1	v_2	$U(\cdot)$
A	0	1	1
B	10	−1	9
C	−10	0	−10

	v_1	v_2	$U(\cdot)$
A	0	100	100
B	10	−100	−90
C	−10	0	−10

	v_1	v_2	$U(\cdot)$
A	0	1	1
B	1	−1	0
C	−1	0	−1

	v_1	v_2	$U(\cdot)$
A	0	1	1
B	10	−1	9
C	−10	0	−10

	v_1	v_2	$U(\cdot)$
A	0	100	100
B	10	−100	−90
C	−10	0	−10

Mary $\quad$ seashore $\succ_M$ museums $\succ_M$ camping

Sam $\quad$ camping $\succ_S$ museums $\succ_S$ seashore

- The seashore is the only alternative that Mary finds bearable, although she feels more negative about going to the mountains than to the museums.
- Each choice is fine with Sam, although he would much prefer going to the mountains.

	Mary	Sam	Total
Seashore	20		
Museums	10		
Mountains	9		

	Mary	Sam	Total
Seashore	20	86	
Museums	10	93	
Mountains	9	100	

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

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Seashore	20	86	106
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For Mary, the difference between the seashore and the mountains crosses the threshold between the bearable and the intolerable. She feels that her “right to an emotionally recuperative vacation” will be violated by following a utilitarian scheme.

	Mary	Sam	Total
Seashore	200	86	286
Museums	100	93	190
Mountains	90	100	190

Mary: My preferences are so intense in comparison with yours that my scale should range between 0 and 1,000, if yours range between 0 and 100.

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

Sam: You think that my preferences are rather weak, but the fact is I feel things quite deeply. I have been brought up in a culture very different from yours and have been trained to avoid emotional outbursts...But I have strong feelings all the same.

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
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Sam: I do not think that extra weight should be given in a utilitarian calculation to those who are capable of more intense preferences.

- Is Mary's preference for the seashore *really* stronger than Sam's for the mountains? Or, is Mary just a more vocal person?

- If some people's preferences are in fact stronger than others', how could we *know* this?

- Does it make any more sense to compare Sam's preferences with Mary's than it does to compare a dog's preference for steak bones with a horse's preference for oats?

- Even if we answer all these questions affirmatively, is it morally proper to respond to such information in making social choices?

Can’t we just wait for psychologists to develop an adequate theory of emotions?

Don’t we make interpersonal comparisons all the time?

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Don’t we make interpersonal comparisons all the time?

Is there more to emotions than our display of them?

Harsanyi’s Theorem

Assume that there is a finite number of citizens ($N = \{1, \ldots, n\}$), and a finite set of social states X.

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Assume that there is a *Planner*.

- The planner's utility function matches the social utility function
- If the Planner is a citizen, he is required to have two (but not necessarily different) preference orderings — his personal ordering and his moral ordering.

Individual and Social Rationality Each citizen and the Planner have a ranking $\succeq_1, \succeq_2, \ldots, \succeq_n, \succeq$ over $\mathcal{L}(X)$ (the set of lotteries over the social states X) satisfying the Von Neumann-Morgenstern axioms.

Individual and Social Rationality Each citizen and the Planner have a ranking $\succeq_1, \succeq_2, \ldots, \succeq_n, \succeq$ over $\mathcal{L}(X)$ (the set of lotteries over the social states X) satisfying the Von Neumann-Morgenstern axioms.

- Each citizen's preference is represented by a linear utility function u_i
- The Planner's preference is represented by a linear utility function u
- Assume that all the citizens use 0 to 1 utility scales.
- Assume that 0 is the lowest utility scale for the Planner.

Strong Pareto

(P1) For each L, L' if $L \sim_i L'$ for all $i \in N$, then $L \sim L'$

(P2) For each L, L' if $L \succeq_i L'$ for all $i \in N$ and $L \succ_j L'$ for some $j \in N$, then $L \succ L'$

Each lottery L is associated with a vector of real numbers, $(u_i(L), \ldots, u_n(L)) \in \Re^n$. That is, the sequence of utility values of L for each agent.

Each lottery L is associated with a vector of real numbers, $(u_i(L), \ldots, u_n(L)) \in \mathfrak{R}^n$. That is, the sequence of utility values of L for each agent.

Defined the following two sets:

$$\mathcal{R}^n = \{(r_1, \ldots, r_n) \in \mathfrak{R}^n \mid \text{ there is a } L \in \mathcal{L} \text{ such that for all } i = 1, \ldots, n, \, u_i(L) = r_i\}$$

and

$$\mathcal{R} = \{r \in \mathfrak{R} \mid \text{there is a } L \in \mathcal{L} \text{ such that } u(L) = r\}$$

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Define a function $f : \mathcal{R}^n \to \mathcal{R}$ as follows: for all $(r_1, \ldots, r_n)$, let $f(r_1, \ldots, r_n) = r$ where $r = u(L)$ with L a lottery such that $(u_1(L), \ldots, u_n(L)) = (r_1, \ldots, r_n)$.

Equity

(E) All agents should be treated equally by the Planner. Formally, this means that $f(r_1, \ldots, r_n) = f(r'_1, \ldots, r'_n)$ when there is a permutation $\pi : N \to N$ such that for each $i = 1, \ldots, n$, $r'_i = r_{\pi(i)}$.

Harsanyi's Theorem For all $(r_1, \ldots, r_n) \in \mathcal{R}^n$, $f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$.

Observation. The function f is well-defined.

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Proof. Suppose that $L, L' \in \mathcal{L}$ such that $(u_1(L), \ldots, u_n(L)) = (u_1(L'), \ldots, u_n(L'))$. Then, for all $i \in N$, i is indifferent between L and L' (i.e., $L \sim_i L'$). Then, by axiom $P1$, we have $L \sim L'$. Thus, $u(L) = u(L')$; and so, f is well-defined.

For each $i \in N$ and $L \in \mathcal{L}$, we have $0 \leq u_i(L) \leq 1$.

For each $i \in N$, let $e_i = (0, 0, \ldots, 1, \ldots, 0)$ (where there is a 1 in the ith position and 0 everywhere else).

This corresponds to a situation in which a single agent gets her most preferred outcome while all the other agents get their least-preferred outcome.

Lemma. For each $i, j \in N$, $f(e_i) = f(e_j)$

Lemma. For all $a \in \mathfrak{R}$, $af(r_1, \ldots, r_n) = f(ar_1, \ldots, ar_n)$.

Let L be the lottery such that for each $i \in N$, $u_i(L) = r_i$. Consider the lottery $L' = [L : a, \mathbf{0} : (1 - a)]$, where $\mathbf{0}$ is the lottery in which everyone gets their lowest-ranked outcome.

Then, for each $i \in N$, $u_i(\mathbf{0}) = 0$. Furthermore, by the Pareto principle $P1$, we must have $u(\mathbf{0}) = 0$.

Then, for all $i \in N$, we have

1. $u_i(L') = au_i(L) + (1-a)u_i(\mathbf{0}) = au_i(L) = ar_i$; and
2. $u(L') = au(L) + (1-a)u(\mathbf{0}) = au(L)$

$$af(r_1, \ldots, r_n) \quad = \quad au(L) \qquad \text{(definition of } f\text{)}$$

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$$
\begin{aligned}
af(r_1, \ldots, r_n) &= au(L) && \text{(definition of } f\text{)} \\
&= u(L') && \text{(item 2.)}
\end{aligned}
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$$
\begin{array}{rcll}
af(r_1,\ldots,r_n) & = & au(L) & \text{(definition of } f\text{)} \\
 & = & u(L') & \text{(item 2.)} \\
 & = & f(u_1(L'),\ldots,u_n(L')) & \text{(definition of } f\text{)}
\end{array}
$$

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$$\begin{array}{rcll} af(r_1, \ldots, r_n) & = & au(L) & \text{(definition of } f\text{)} \\ & = & u(L') & \text{(item 2.)} \\ & = & f(u_1(L'), \ldots, u_n(L')) & \text{(definition of } f\text{)} \\ & = & f(ar_1, \ldots ar_n) & \text{(item 1.)} \end{array}$$

Theorem. For all $(r_1, \ldots, r_n) \in \mathcal{R}^n, f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$.

Consider a lottery L such that for all $i \in N$, $u_i(L) = r_i$. Consider lotteries L_i such that $u_i(L_i) = r_i$ and for all $j \neq i$, $u_j(L_i) = 0$. Consider the lottery $L' = [L_1 : 1/n, \ldots, L_n : 1/n]$.

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- $u_i(L') = \sum_{k=1}^{n} \frac{1}{n} u_i(L_k) = \frac{1}{n} u_i(L_i) = \frac{1}{n} r_i$.
- $f(0, \dots, r_k, \dots, 0) = r_k f(0, \dots, 1, \dots, 0) = r_k$

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	1	2	P
L_1	r_1	0	$f(r_1, 0) = r_1 f(1, 0)$
L_2	0	r_2	$f(0, r_2) = r_2 f(0, 1)$
L'	$\frac{1}{2}u(L_1) + \frac{1}{2}u(L_2) = \frac{1}{2}r_1$	$\frac{1}{2}u(L_1) + \frac{1}{2}u(L_2) = \frac{1}{2}r_1$	$f(\frac{1}{2}r_1, \frac{1}{2}r_2)$

$$\frac{1}{2} f(r_1, r_2) = f(\frac{1}{2} r_1, \frac{1}{2} r_2) = u(L') = \frac{1}{2} u(L_1) + \frac{1}{2} u(L_2) = \frac{1}{2} r_1 f(1, 0) + \frac{1}{2} r_2 f(0, 1)$$

$$u(L') \quad = \quad \textstyle\sum_{k=1}^{n} \frac{1}{n}\, u(L_k)$$

$$
\begin{aligned}
u(L') &= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, u(L_k) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, f(u_1(L_k), \ldots, u_k(L_k), \ldots, u_n(L_k))
\end{aligned}
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&= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, r_k
\end{aligned}
$$

$$u(L') \quad = \quad f(u_1(L'), \ldots, u_n(L'))$$

$$
\begin{aligned}
u(L') &= f(u_1(L'), \ldots, u_n(L')) \\
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\end{aligned}
$$

$$
\begin{aligned}
u(L') &= f(u_1(L'), \ldots, u_n(L')) \\
&= f(\tfrac{1}{n}\, r_1, \ldots, \tfrac{1}{n} r_n) \\
&= \tfrac{1}{n} f(r_1, \ldots, r_n)
\end{aligned}
$$

Thus,

$$\frac{1}{n} f(r_1, \ldots, r_k) = u(L') = \sum_{k=1}^{n} \frac{1}{n}\, r_k = \frac{1}{n}\, \sum_{k=1}^{n} r_k$$

Hence, $f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$, as desired.

For 2 citizens, Harsanyi's Theorem require the existence of the following vectors of utilities:

$$(0,0) \quad (0,1) \quad (1,0) \quad (u_1,0) \quad (0,u_2) \quad (u_1,u_2)$$

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Problem. None of Harsanyi's conditions guarantee the existence of this social outcomes.

Suppose the problem is to give a scholarship to *exactly* one of the citizens.

- $(1, 0)$: give the scholarship to citizen 1
- $(0, 1)$: give the scholarship to citizen 2

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- $(1, 0)$: give the scholarship to citizen 1
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- What is the outcome $(0, 0)$?

Distributable Goods Assumption

For every vector of numbers $(u_1, \ldots, u_n)$ with $0 \leq u_i \leq 1$, there is at least one social option for which the distribution of citizens' utilities equals the vector in question.

A distributable good is one, such as food, health, education, talent, friendship, for which all distributions throughout society are at least logically possible.

Problem: Philosophers also look to social choice theory for help in resolving problems in which interests conflict-situations, for example, in which citizens gain only at the expense of others, or ones in which the citizens envy each other, or prefer to sacrifice for each other. These are situations in which we cannot count on the distributable goods assumption to hold.

- A defense of the theorem must argue either that a “true” representation of the citizens’ preferences will give rise to the appropriate vectors or that there is a set of “background” options sufficiently rich to support the same vectors, or that certain profiles, such as those in which considerations of envy or altruism are operative, should not be considered.

1. An employer must choose between two equally qualified employees to promote. Assume that everything about their contributions to the firm, their length of service, personal financial needs, and so forth, is the same. The employer summons both employees to her office for separate conversations. The first is an impassive type who allows that he would be pleased to be promoted. The second, on the other hand, effusively tells the employer how long he has hoped for the promotion, etc.

1. An employer must choose between two equally qualified employees to promote. Assume that everything about their contributions to the firm, their length of service, personal financial needs, and so forth, is the same. The employer summons both employees to her office for separate conversations. The first is an impassive type who allows that he would be pleased to be promoted. The second, on the other hand, effusively tells the employer how long he has hoped for the promotion, etc. The employer promotes the second employee explaining that "it meant so much more to the second"...

2. A politician must decide whether to demolish a block of old houses to make room for a new library. The residents of the houses are old and feeble, and the sponsors of the library are young and quite vocal. Both send delegates to speak to the politician. The politician finds it politically expedient to favor the young.

Those that believe in interpersonal comparison of utilities will grant that the two cases have been correctly described: The employer weighed the utilities of her two employees and the politician simply responded to political pressure.

Those who are skeptical about interpersonal comparisons of utility, will argue that in both cases the decision maker is simply behaving in accordance with cultural conditioning to respond in certain ways to the actions of others...the second employee's effusiveness is just as much a form of pressure as the political activists'.

- interpersonal comparison of utility *levels*
- interpersonal comparison of utility *increments*

Harsanyi's social welfare function deals with incremental utilities and ignores utility levels.

- The ranking of x and y in terms of sums is preserved if adding (the same or different) numbers to both x and y. Adding these numbers is tantamount to changing the zero points of the citizens' utilities.
- Harsanyi's social welfare function does respond to changes in the units used to measure utility increments.

Some social choice methods respond only to changes in the utility origins, these presuppose the interpersonal comparison of *utility origins*.

Some social choice methods respond only to change in utility units and presuppose interpersonal comparison of utility units.

Some social choice methods respond to changes in both utility origins and units and presupposes interpersonal comparison of both.