PHIL309P
Philosophy, Politics and Economics

Eric Pacuit
Department of Philosophy
University of Maryland
pacuit.org

Notation

- N is a finite set of voters (assume that $N = \{1, 2, 3, \ldots, n\}$)
- C is a (typically finite) set of alternatives, or candidates
- A relation on C is a **linear order** if it is transitive, irreflexive, and complete (hence, acyclic), also called a **ranking**
- $L(X)$ is the set of all linear orders over the set X
- $O(X)$ is the set of all reflexive and transitive relations over the set X (ties are allowed)

Profiles

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So, $a\mathbf{P}_i b$ means that voter i strictly prefers candidate a to b, or a is ranked above b.

For instance, let $V = \{1, 2, 3, 4\}$ and $C = \{a, b, c, d\}$. Then, an example of a profile is:

1	2	3	4
a	a	b	c
b	c	a	b
c	b	c	a

- A **profile** for the set of voters V is a sequence of (linear) orders over C, denoted $\mathbf{P} = (P_1, \ldots, P_n)$.
  - Note that unlike V or C, which are sets (order of elements **does not** matter), $\mathbf{P}$ is a *tuple* of different rankings (i.e., the order of the rankings **does** matter!).

- If $|C| = n$ and $|V| = m$, we call $\mathbf{P}$ a (n, m)**-profile**.

- $L(C)^V$ is the set of all **profiles or linear orders** for n voters (similarly for $O(C)^V$)

Voting Method

A **voting method** is a function $f : L(C)^V \to \wp(C) \setminus \{\varnothing\}$.

A voting method is **resolute** if for all profiles $\mathbf{P}$, $|f(\mathbf{P})| = 1$.

Anonymous Profiles

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2	5	3	5
a	a	b	c
b	c	a	b
c	b	c	a

Majoritarianism

When there are only **two** candidates A and B, then all (reasonable) voting methods give the same results:

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Yes. We will look at two arguments: A procedural justification and an epistemic justification.

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Results are more mixed: Consider our previous definition of majority rule....we only defined it between two options! Can we generalize for $|C| > 2$?

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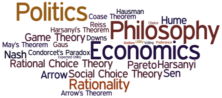
Majority Rule



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Is this a good generalization? What problems might be run into?

- If might not return a winner, especially as C grows!
- Tyranny of the majority: A candidate with 51% of the vote may be ranked last by 49% of the voters, while another candidate is ranked 1st or 2nd by 100% of the voters.

Positional scoring rules

Suppose $\langle s_1, s_2, \ldots, s_m \rangle$ is a vector of numbers, called a **scoring vector**, where for each $l = 1, \ldots, m-1$, $s_l \geq s_{l+1}$.

The **score of** $x \in C$ **given** P is $score(P, x) = s_r$ where r is the rank of x in P.

For each profile $\mathbf{P}$ and $x \in C$, let $score(\mathbf{P}, x) = \sum_{i=1}^{n} score(\mathbf{P}_i, x)$.

A voting method f is a positional scoring rule for a scoring vector $\vec{s}$ provided that for all $\mathbf{P} \in L(C)^V$, $f(\mathbf{P}) = \text{argmax}_{x \in C} score(\mathbf{P}, x)$.

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Borda: the positional scoring rule for $\langle n-1, n-2, \ldots, 1, 0 \rangle$.

Plurality: the positional scoring rule for $\langle 1, 0, \ldots, 0 \rangle$.

# voters	7	5	4	3
	A	B	D	C
	B	C	B	D
	C	D	C	A
	D	A	A	B

# voters	7	5	4	3
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Plurality winners A
Plurality scores $A:7,\ B:5,\ C:3,\ D:4$

# voters	7	5	4	3
	A	B	D	C
	B	C	B	D
	C	D	C	A
	D	A	A	B

Plurality winners	A
Plurality scores	$A:7,\ B:5,\ C:3,\ D:4$
Borda winners	B
Borda scores	$A:24,\ B:37,\ C:30,\ D:23$

# voters	7	5	4	3
	A	B	D	C
	B	C	B	D
	C	D	C	A
	D	A	A	B

There is no absolute majority winner. Which candidate(s) is(are) the ”closest” to the majority winner?

Let’s start with an example involving the voting method known as “**Ranked Choice Voting**,” “**Instant Runoff**,” or “**Hare System**.”

This is widely used in Australia and is promoted in the U.S. by FairVote.org and the anti-corruption campaign RepresentUs.

FairVote
FOR A MORE PERFECT UNION

Hare

Iteratively remove all candidates with the fewest number of voters who rank them first, until there is a candidate who is a majority winner. If, at some stage of the removal process, all remaining candidates have the same number of voters who rank them first (so all candidates would be removed), then all remaining candidates are selected as winners.

Plurality with Runoff

Calculate the plurality score for each candidate—the number of voters who rank the candidate first. If there are 2 or more candidates with the highest plurality score, remove all other candidates and select the Plurality winners from the remaining candidates. If there is one candidate with the highest plurality score, remove all candidates except the candidates with the highest or second-highest plurality score, and select the Plurality winners from the remaining candidates.

Coombs

Iteratively remove all candidates with the most number of voters who rank them last, until there is a candidate who is a majority winner. If, at some stage of the removal process, all remaining candidates have the same number voters who rank them last (so all candidates would be removed), then all remaining candidates are selected as winners.

Baldwin

Iteratively remove all candidates with the smallest Borda score, until there is a single candidate remaining. If, at some stage of the removal process, all remaining candidates have the same Borda score (so all candidates would be removed), then all remaining candidates are selected as winners.

Nanson

Rather than removing candidates with the lowest Borda score, the next two methods remove all candidates who have a Borda score below the average Borda score for all candidates. **Nanson** iteratively removes all candidates whose Borda score is strictly smaller than the average Borda score (of the candidates remaining at that stage), until one candidate remains.

If, at some stage of the removal process, all remaining candidates have the same Borda score (so all candidates would be removed), then all remaining candidates are selected as winners.

# voters	7	5	4	3
	A	B	D	C
	B	C	B	D
	C	D	C	A
	D	A	A	B

PluralityWRunoff winners	A
Hare winners	D
Coombs winners	B
Nanson winners	B
Baldwin winners	A

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections.

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	B	C	D	**B**
	C	**B**	C	D
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# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections. B is ranked second, D is ranked third, and A is ranked last.

$$C >_M B >_M D >_M A$$

The Majority Relation

Suppose that X and Y are candidates and P_i represents voter i's *strict preference*.

$\mathbf{N}(X\ P\ Y) = |\{i \mid X\ P_i\ Y\}|$
"the number of voters that rank X strictly above Y"

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X is a **Condorcet winner** if X beats every other candidate in an head-to-head election: there is no candidate Y such that $Y >_M X$

X is a **Condorcet loser** if X loses to every other candidate in an head-to-head elections: there is no candidate Y such that, $X >_M Y$

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

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Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B?

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Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B? Yes

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B? Yes
- Does the group prefer B over C? Yes

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B? Yes
- Does the group prefer B over C? Yes
- Does the group prefer A over C? No

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

The majority relation $>_M$ is **not** transitive!

There is a **Condorcet cycle**: $A >_M B >_M C >_M A$

How bad is this?

- Final decisions are extremely sensitive to institutional features such as who can set the agenda, arbitrary time limits place on deliberation, who is permitted to make motions, etc.

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- Is there *empirical evidence* that Condorcet cycles have shown up in real elections?

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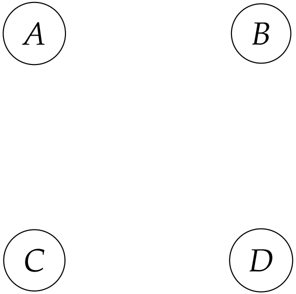
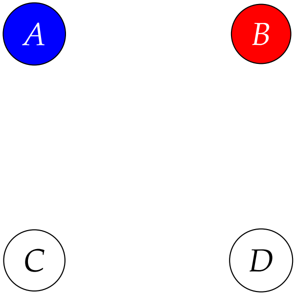
- How *likely* is a Condorcet cycle?

A voting method is **Condorcet consistent** if it selects the Condorcet winner if it exists.

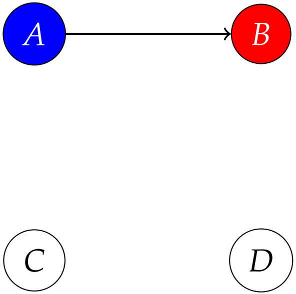
7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B



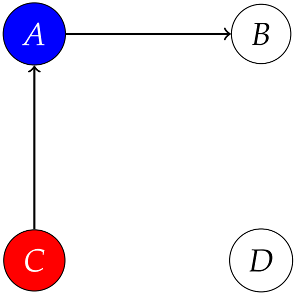
7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B



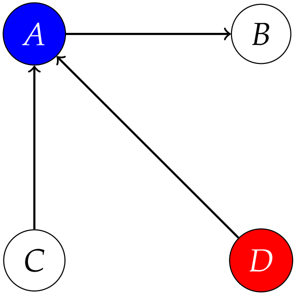
7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B



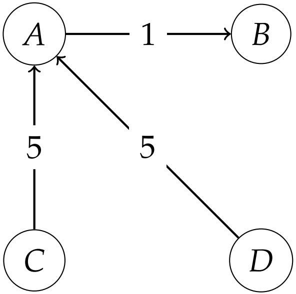
7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B



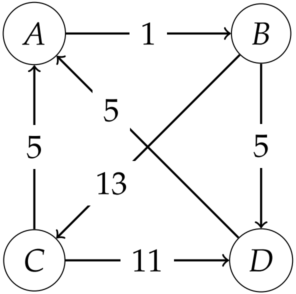
7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B



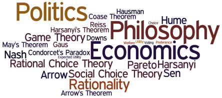
7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B



7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B



Condorcet



The **Condorcet winner** in a profile $\mathbf{P}$ is a candidate $x \in C$ that is the maximum of the majority ordering, i.e., for all $y \in C$, if $x \neq y$, then $x >^M_{\mathbf{P}} y$. The Condorcet voting method is:

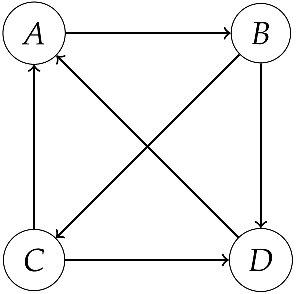
$$Condorcet(\mathbf{P}) = \begin{cases} \{x\} & \text{if } x \text{ is the Condorcet winner in } \mathbf{P} \\ C & \text{if there is no Condorcet winner.} \end{cases}$$

Copeland

For each $\mathbf{P}$ and $x \in C$, let $wl_{\mathbf{P}}(x) = |\{z \mid Net_{\mathbf{P}}(x,z) > 0\}| - |\{z \mid Net_{\mathbf{P}}(z,x) > 0\}|$.

$Copeland(\mathbf{P}) = \text{argmax}_{x \in C}(wl_{\mathbf{P}}(x))$.

7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B

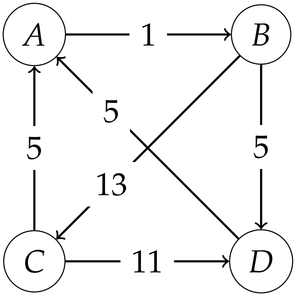


MaxMin

For each $\mathbf{P}$ and $x \in C$, let $supp(x, \mathbf{P}) = \max(\{\mathbf{N_P}(y, x) \mid y \in C, y \neq x\})$.

$MinMax(\mathbf{P}) = \text{argmin}_{x \in C}(supp(x, \mathbf{P}))$.

7	5	4	3
A	B	D	C
B	C	B	D
C	D	C	A
D	A	A	B



Condorcet winners	A, B, C, D
Copeland winners	B, C
MinMax winners	B
Beatpath winners	B

Voting Methods Tutorial

For n candidates and m voters, there are $n!^m$ profiles.

candidates	voters	number of profiles
3	3	216
3	4	1296
3	5	7776
3	6	46656
3	7	279936
3	8	1679616
4	3	13824
4	4	331776
4	5	7962624
4	6	191102976
4	7	4586471424
4	8	110075314176
5	3	1728000
5	4	207360000
5	5	24883200000
5	6	2985984000000
5	7	358318080000000
5	8	42998169600000000

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5	4	207360000
5	5	24883200000
5	6	2985984000000
5	7	358318080000000
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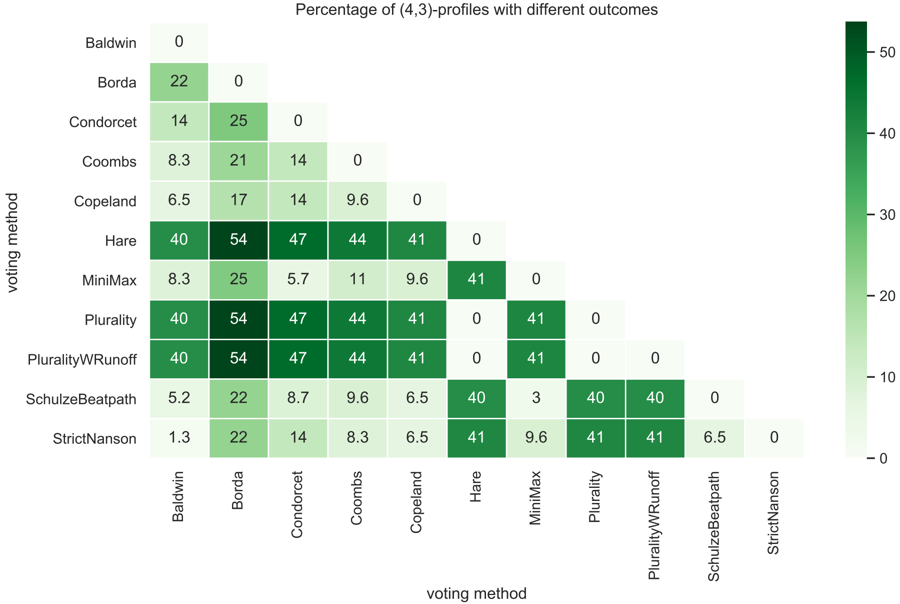
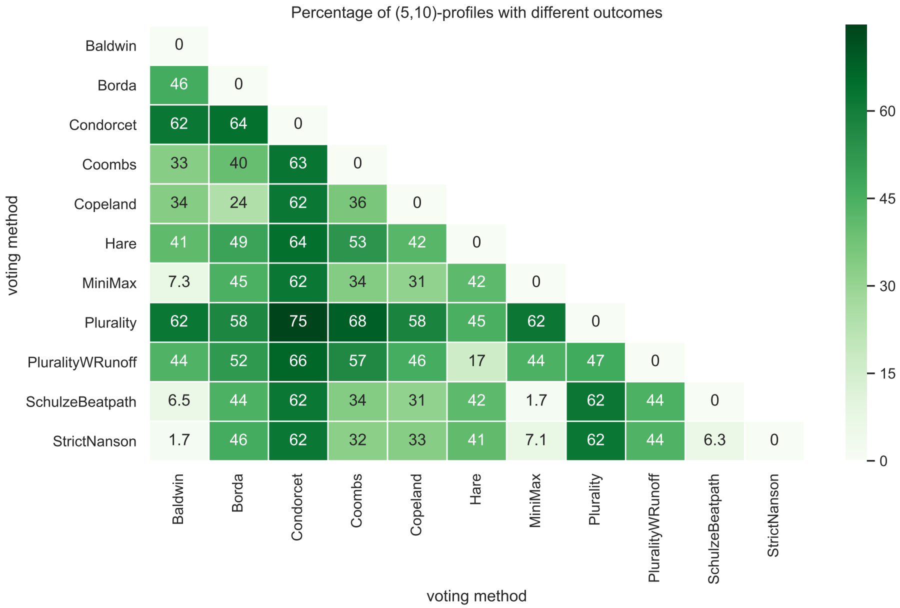

Percentage of (4,3)-profiles with different outcomes
Baldwin
Borda
Condorcet
Coombs
Copeland
Hare
MiniMax
Plurality
PluralityWRunoff
SchulzeBeatpath
StrictNanson
voting method
0
22 0
14 25 0
8.3 21 14 0
6.5 17 14 9.6 0
40 54 47 44 41 0
8.3 25 5.7 11 9.6 41 0
40 54 47 44 41 0 41 0
40 54 47 44 41 0 41 0 0
5.2 22 8.7 9.6 6.5 40 3 40 40 0
1.3 22 14 8.3 6.5 41 9.6 41 41 6.5 0
50
40
30
20
10
0


Percentage of (5,10)-profiles with different outcomes
Baldwin 0
Borda 46 0
Condorcet 62 64 0
Coombs 33 40 63 0
Copeland 34 24 62 36 0
Hare 41 49 64 53 42 0
MiniMax 7.3 45 62 34 31 42 0
Plurality 62 58 75 68 58 45 62 0
PluralityWRunoff 44 52 66 57 46 17 44 47 0
SchulzeBeatpath 6.5 44 62 34 31 42 1.7 62 44 0
StrictNanson 1.7 46 62 32 33 41 7.1 62 44 6.3 0
Baldwin
Borda
Condorcet
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Copeland
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MiniMax
Plurality
PluralityWRunoff
SchulzeBeatpath
StrictNanson
voting method
voting method
60
45
30
15
0

Should we select a Condorcet winner (when one exists)?

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
2	A	A	B	B	C	C
1	B	C	A	C	A	B
0	C	B	C	A	B	A

$$BS(A) = 2 \times 31 + 1 \times 39 + 0 \times 11 = 101$$
$$BS(B) = 2 \times 39 + 1 \times 31 + 0 \times 11 = 109$$
$$BS(C) = 2 \times 11 + 1 \times 11 + 0 \times 59 = 33$$

$$B >_{BC} A >_{BC} C$$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
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	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

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# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

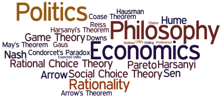
Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox



# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$

$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$

$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

$Score(A) > Score(B) \Rightarrow 31s_2 + 39s_1 > 39s_2 + 31s_1 \Rightarrow s_1 > s_2$

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Theorem (Fishburn 1974). For all $m \geq 3$, there is some voting situation with a Condorcet winner such that every scoring rule will have at least $m - 2$ candidates with a greater score than the Condorcet winner.

P. Fishburn. *Paradoxes of Voting*. The American Political Science Review, 68:2, pgs. 537 - 546, 1974.

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

# voters	30	1	29	10	10	1
2	A	A	B	B	C	C
1	B	C	A	C	A	B
0	C	B	C	A	B	A

$$BS(A) = 2 \times 31 + 1 \times 39 + 0 \times 11 = 101$$
$$BS(B) = 2 \times 39 + 1 \times 31 + 0 \times 11 = 109$$
$$BS(C) = 2 \times 11 + 1 \times 11 + 0 \times 59 = 33$$

$$B >_{BC} A >_{BC} C$$

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_{M} B >_{M} C$

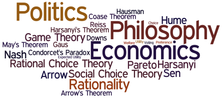
# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$$B >_{BC} A >_{BC} C \qquad\qquad A >_M B >_M C$$

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet Triples



G_1	G_2	G_3
A	B	C
B	C	A
C	A	B

G_1	G_2	G_3
A	C	B
C	B	A
B	A	C

If $G_1 = G_2 = G_3$, then this group of voters "cancel out" each other's votes

Saari's argument

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

Saari's argument

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

10	10	10
A	B	C
B	C	A
C	A	B

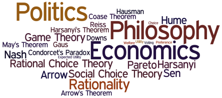
Saari’s argument

# voters	20	1	29	0	0	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

10	10	10
A	B	C
B	C	A
C	A	B

1	1	1
A	C	B
C	B	A
B	A	C

Saari's argument



# voters	20	0	28	0	0	0
	A		B			
	B		A			
	C		C			

10	10	10
A	B	C
B	C	A
C	A	B

1	1	1
A	C	B
C	B	A
B	A	C

Is the Condorcet winner the “best” choice?

# voters	47	47	3	3
	A	B	C	C
	C	C	A	B
	B	A	B	A

C is the Condorcet winner

Is the Condorcet winner the "best" choice?

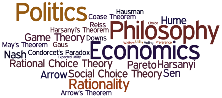
# voters	47	47	3	3
	A	B	C	C
	C	C	A	B
	B	A	B	A

C is the Condorcet winner; however, it seems that supporters of the main rivals A and B would rather see C win than their candidate's principal opponent, but this does not mean that there is "positive support" for C.

Further Investigation

- W. Poundstone, *Gaming the Vote: Why Elections Aren't Fair (and What We Can Do About It)*, Hill and Wang, 2009
- EP, Voting Methods (Stanford Encyclopedia of Philosophy)
- C. List, Social Choice Theory (Stanford Encyclopedia of Philosophy)
- M. Morreau, Arrow's Theorem (Stanford Encyclopedia of Philosophy)

Further Investigation



- https://www.electology.org
- http://www.fairvote.org
- http://rangevoting.org
- https://www.opavote.com
- http://www.preflib.org

There are many different voting methods

Many different electoral methods: Plurality, Borda Count, Antiplurality/Veto, and k-approval; Plurality with Runoff; Single Transferable Vote (STV)/Hare; Approval Voting; Cup Rule/Voting Trees; Copeland; Banks; Slater Rule; Schwartz Rule; the Condorcet rule; Maximin/Simpson, Kemeny; Ranked Pairs/Tideman; Bucklin Method; Dodgson Method; Young's Method; Majority Judgment; Cumulative Voting; Range/Score Voting; …

Choosing how to choose

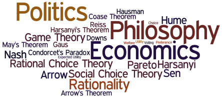
Pragmatic considerations: Is the procedure easy to use? Is it legal? The importance of ease of use should not be underestimated: Despite its many flaws, plurality rule is, by far, the most commonly used method.

Behavioral considerations: Do the different procedures *really* lead to different outcomes in practice?

Information required from the voters: What type of information do the ballots convey? I.e., Choosing a single alternative, linearly rank all the candidates, report something about the "intensity" of preference.

Axiomatics: Characterize the different voting methods in terms of normative principles of group decision making.

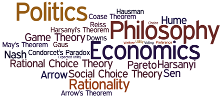
Principles of group decision making



Principles of group decision making

- **Condorcet Condition**: Always choose the candidate that beats every other candidate in head-to-head elections.

Principles of group decision making



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- **Unanimity (Pareto)**: If *everyone* ranks A above B, then B should not win the election.

Principles of group decision making

- **Condorcet Condition**: Always choose the candidate that beats every other candidate in head-to-head elections.
- **Unanimity (Pareto)**: If *everyone* ranks A above B, then B should not win the election.
- **Anonymity**: The names of the voters do not matter (if two voters swap votes, then the outcome is unaffected).

Monotonicity

A candidate receiving more “support” shouldn’t maker her worse off.

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A candidate receiving more "support" shouldn't maker her worse off.

More-is-Less Paradox: If a candidate C is elected under a given a profile of rankings of the competing candidates, it is possible that, *ceteris paribus*, C may not be elected if some voter(s) raise C in their rankings.

P. Fishburn and S. Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine (1983).

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
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	A	C	B	A
	B	A	C	B
	C	B	A	C

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: *A*

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: *A*

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: *A*

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Winner: *C*

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	**B**
	B	A	C	**A**
	C	B	A	**C**

Winner: *A*

# voters	6	5	4	2
	A	C	B	**A**
	B	A	C	**B**
	C	B	A	**C**

Winner: *C*

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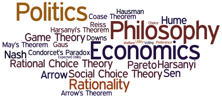
- **Twin Paradox**: A voter may obtain a less preferable outcome if his "twin" (a voter with the exact same ranking) decides to participate in the election.



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No-Show Paradox: A voter may obtain a more preferable outcome if he decides not to participate in an election than, *ceteris paribus*, if he decides to participate in the election.

- **Twin Paradox**: A voter may obtain a less preferable outcome if his "twin" (a voter with the exact same ranking) decides to participate in the election.
- **Truncation Paradox**: A voter may obtain a more preferable outcome if, *ceteris paribus*, he only reveals part of his ranking of the candidates.

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: C

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: C

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: C

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: *C*

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: *B*

Twin Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: *C*

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: *B*

Failures of Monotonicity

Example: Burlington, VT 2009 Mayoral Race
(rangevoting.org/Burlington.html)

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D. Felsenthal and N. Tideman. *Varieties of Failure of Monotonicity and Participation under Five Voting Methods*. Theory and Decision, 75, pgs. 59 - 77, 2013.

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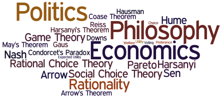
D. Felsenthal and N. Tideman. *Varieties of Failure of Monotonicity and Participation under Five Voting Methods*. Theory and Decision, 75, pgs. 59 - 77, 2013.

Theorem (Moulin). If there are four or more candidates, then every Condorcet consistent voting methods is susceptible to the No-Show paradox.

H. Moulin. *Condorcet's Principle Implies the No Show Paradox*. Journal of Economic Theory, 45, pgs. 53 - 64, 1988.

Principles



Condorcet: Elect the Condorcet winner whenever it exists.

Monotonicity: More support should never hurt a candidate.

Participation: It should never be in a voter's best interests not to vote.

Multiple-Districts: If a candidate wins in each district, then that candidate should also win when the districts are merged.

More Principles

Pareto: Never elect a candidate if another candidate is strictly preferred by all voters.

Anonymity: The outcome does not depend on the names of the voters.

Neutrality: The outcome does not depend on the names of the candidates.

Universal Domain: The voters are free to rank the candidates (or grade the candidates) in any way they want.

What are the relationships between these principles? Is there a procedure that satisfies *all* of them?

What are the relationships between these principles? Is there a procedure that satisfies *all* of them?

A few observations:

- Condorcet winners may not exist.
- No positional scoring method satisfies the Condorcet Principle.
- The Condorcet and Participation principles cannot be jointly satisfied.

Axiomatics

"When a set of axioms regarding social choice can all be simultaneously satisfied, there may be several possible procedures that work, among which we have to choose.

A. Sen. *The Possibility of Social Choice*. The American Economic Review, 89:3, pgs. 349 - 378, 1999 (reprint of his Nobel lecture).

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"When a set of axioms regarding social choice can all be simultaneously satisfied, there may be several possible procedures that work, among which we have to choose. In order to choose between different possibilities through the use of discriminating axioms, we have to introduce *further* axioms, until only and only one possible procedure remains.

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"When a set of axioms regarding social choice can all be simultaneously satisfied, there may be several possible procedures that work, among which we have to choose. In order to choose between different possibilities through the use of discriminating axioms, we have to introduce *further* axioms, until only and only one possible procedure remains. This is something of an exercise in brinkmanship. We have to go on and on cutting alternative possibilities, moving—implicitly—*towards* an impossibility, but then stop just before all possibilities are eliminated, to wit, when one and only one options remains." (pg. 354)

A. Sen. *The Possibility of Social Choice*. The American Economic Review, 89:3, pgs. 349 - 378, 1999 (reprint of his Nobel lecture).

The Social Choice Model

Notation

- N is a finite set of voters (assume that $N = \{1, 2, 3, \ldots, n\}$)
- X is a (typically finite) set of alternatives, or candidates
- A relation on X is a linear order if it is transitive, irreflexive, and complete (hence, acyclic)
- $L(X)$ is the set of all linear orders over the set X
- $O(X)$ is the set of all reflexive and transitive relations over the set X

Notation

- A **profile** for the set of voters N is a sequence of (linear) orders over X, denoted $\mathbf{R} = (R_1, \ldots, R_n)$.

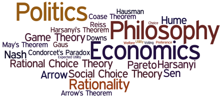
- $L(X)^n$ is the set of all **profiles** for n voters (similarly for $O(X)^n$)

- For a profile $\mathbf{R} = (R_1, \ldots, R_n) \in O(X)^n$, let $\mathbf{N}_{\mathbf{R}}(A\ P\ B) = \{i \mid A\ P_i\ B\}$ be the set of voters that rank A above B (similarly for $\mathbf{N}_{\mathbf{R}}(A\ I\ B)$ and $\mathbf{N}_{\mathbf{R}}(B\ P\ A)$)

Preference Aggregation Methods

Social Welfare Function: $F : \mathcal{D} \to L(X)$, where $\mathcal{D} \subseteq L(X)^n$

Preference Aggregation Methods

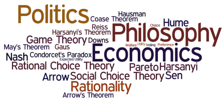


Social Welfare Function: $F : \mathcal{D} \to L(X)$, where $\mathcal{D} \subseteq L(X)^n$

Comments

- $\mathcal{D}$ is the *domain* of the function: it is the set of all possible profiles
- Aggregation methods are *decisive*: every profile $\mathbf{R}$ in the domain is associated with exactly one ordering over the candidates
- The range of the function is $L(X)$: the social ordering is assumed to be a linear order
- Tie-breaking rules are built into the definition of a preference aggregation function

Preference Aggregation Methods



Social Welfare Function: $F : \mathcal{D} \rightarrow L(X)$, where $\mathcal{D} \subseteq L(X)^n$

Variants

- Social Choice Function: $F : \mathcal{D} \rightarrow \wp(X) - \emptyset$, where $\mathcal{D} \subseteq L(X)^n$ and $\wp(X)$ is the set of all subsets of X.
- Allow Ties: $F : \mathcal{D} \rightarrow O(X)$ where $O(X)$ is the set of orderings (reflexive and transitive) over X
- Allow Indifference and Ties: $F : \mathcal{D} \rightarrow O(X)$ where $O(X)$ is the set of orderings (reflexive and transitive) over X and $\mathcal{D} \subseteq O(X)^n$

Examples

$Maj(\mathbf{R}) = \ >_M$ where $A >_M B$ iff $|\mathbf{N}_{\mathbf{R}}(A\ P\ B)| > |\mathbf{N}_{\mathbf{R}}(B\ P\ A)|$

(the problem is that $>_M$ may not be transitive (or complete))

Examples

$Maj(\mathbf{R}) = \ >_M$ where $A >_M B$ iff $|\mathbf{N_R}(A\ P\ B)| > |\mathbf{N_R}(B\ P\ A)|$

(*the problem is that* $>_M$ *may not be transitive (or complete)*)

$Borda(\mathbf{R}) \ = \ \geq_{BC}$ where $A \geq_{BC} B$ iff the Borda score of A is greater than the Borda score for B.

(*the problem is that* $\geq_{BC}$ *may not be a linear order*)

Characterizing Majority Rule

When there are only **two** candidates A and B, then all voting methods give the same results

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Majority Rule: A is ranked above (below) B if more (fewer) voters rank A above B than B above A, otherwise A and B are tied.

Characterizing Majority Rule

When there are only **two** candidates A and B, then all voting methods give the same results

Majority Rule: A is ranked above (below) B if more (fewer) voters rank A above B than B above A, otherwise A and B are tied.

When there are only two options, can we argue that majority rule is the "best" procedure?

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

May's Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of n voters and $X = \{A, B\}$ the set of candidates.

Social Welfare Function: $F : O(X)^n \rightarrow O(X)$, where $O(X)$ is the set of orderings over X
(*there are only three possibilities:* $A\ P\ B$, $A\ I\ B$, *or* $B\ P\ A$)

$$F_{Maj}(\mathbf{R}) = \begin{cases} A\ P\ B & \text{if } |\mathbf{N}_{\mathbf{R}}(A\ P\ B)| > |\mathbf{N}_{\mathbf{R}}(B\ P\ A)| \\ A\ I\ B & \text{if } |\mathbf{N}_{\mathbf{R}}(A\ P\ B)| = |\mathbf{N}_{\mathbf{R}}(B\ P\ A)| \\ B\ P\ A & \text{if } |\mathbf{N}_{\mathbf{R}}(B\ P\ A)| > |\mathbf{N}_{\mathbf{R}}(A\ P\ B)| \end{cases}$$

May's Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of n voters and $X = \{A, B\}$ the set of candidates.

Social Welfare Function: $F : \{1, 0, -1\}^n \rightarrow \{1, 0, -1\}$,

where 1 means $A\ P\ B$, 0 means $A\ I\ B$, and -1 means $B\ P\ A$

$$F_{Maj}(\mathbf{v}) = \begin{cases} 1 & \text{if } |\mathbf{N_v}(1)| > |\mathbf{N_v}(-1)| \\ 0 & \text{if } |\mathbf{N_v}(1)| = |\mathbf{N_v}(-1)| \\ -1 & \text{if } |\mathbf{N_v}(-1)| > |\mathbf{N_v}(1)| \end{cases}$$

Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there?

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Suppose that there are two voters and two candidates. How many social choice functions are there? $19,683$

- There are three possible rankings for 2 candidates.
- When there are two voters there are $3^2 = 9$ possible profiles:

$$\{(1,1),(1,0),(1,-1),(0,1),(0,0),(0,-1),(-1,1),(-1,0),(-1,-1)\}$$

- Since there are 9 profiles and 3 rankings, there are $3^9 = 19,683$ possible preference aggregation functions.

May’s Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.

- **Anonymity**: all voters should be treated equally.

- **Neutrality**: all candidates should be treated equally.

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- **Unanimity**: unanimously supported alternatives must be the social outcome.

  If $\mathbf{v} = (v_1, \ldots, v_n)$ with for all $i \in N$, $v_i = x$ then $F(\mathbf{v}) = x$ (for $x \in \{1, 0, -1\}$).

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  $F(v_1, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where $v_i \in \{1, 0, -1\}$ and π is a permutation of the voters.

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- **Neutrality**: all candidates should be treated equally.

  $F(-\mathbf{v}) = -F(\mathbf{v})$ where $-\mathbf{v} = (-v_1, \ldots, -v_n)$.

May's Theorem: Details

- **Positive Responsiveness** (Monotonicity): unidirectional shift in the voters' opinions should help the alternative toward which this shift occurs

If $F(\mathbf{v}) = 0$ or $F(\mathbf{v}) = 1$ and $\mathbf{v} < \mathbf{v}'$, then $F(\mathbf{v}') = 1$
where $\mathbf{v} < \mathbf{v}'$ means for all $i \in N$ $v_i \leq v'_i$ and there is some $i \in N$ with $v_i < v'_i$.

Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there that satisfy anonymity?

Anonymity: all voters should be treated equally.

$F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there that satisfy anonymity? 729

Anonymity: all voters should be treated equally.

$F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

- Imposing anonymity reduces the number of preference aggregation functions.
- If F satisfies anonymity, then $F(1, 0) = F(0, 1)$, $F(1, -1) = F(-1, 1)$ and $F(-1, 0) = F(0, -1)$.
- This means that there are essentially 6 elements of the domain. So, there are $3^6 = 729$ preference aggregation functions.

May’s Theorem: Details

May’s Theorem (1952) A social decision method F satisfies unanimity, neutrality, anonymity and positive responsiveness iff F is majority rule.

Proof Idea

If $(1, 0, -1)$ is assigned 1 or -1 then

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✓ Neutrality implies $(1, 0, -1)$ is assigned -1 or 1
Contradiction.

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Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

- ✓ Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1
- ✓ Anonymity implies $(1, -1, -1)$ is assigned 0 or 1
- ✓ Positive Responsiveness implies $(1, 0, -1)$ is assigned 1
- ✓ Positive Responsiveness implies $(1, 1, -1)$ is assigned 1
  Contradiction.

Other characterizations

G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

Can May's Theorem be generalized to more than 2 candidates?

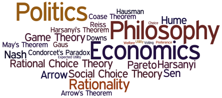
Can May's Theorem be generalized to more than 2 candidates? No!

Spoiler Candidates: Plurality Rule

# voters	49	48	3
	A	B	C
	B	A	B
	C	C	A

Winner: A

Spoiler Candidates: Plurality Rule



# voters	49	48	3
	A	B	C
	B	A	B
	C	C	A

Winner: B

IIA

Independence of Irrelevant Alternatives: If the voters in two different electorates rank A and B in exactly the same way, then A and B should be ranked the same way in both elections.

Failure of IIA: Borda Count

# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

# voters	3	2	2
	A	B	C
	B	C	X
	C	X	A
	X	A	B

Failure of IIA: Borda Count

# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

# voters	3	2	2
	A	B	C
	B	C	X
	C	X	A
	X	A	B

A (15) $>_{BC}$ B (14) $>_{BC}$ C (13) $>_{BC}$ X (0)

Failure of IIA: Borda Count

# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

# voters	3	2	2
3	A	B	C
2	B	C	X
1	C	X	A
0	X	A	B

$A \ (15) >_{BC} B \ (14) >_{BC} C \ (13) >_{BC} X \ (0)$

Failure of IIA: Borda Count

# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

# voters	3	2	2
3	A	B	C
2	B	C	X
1	C	X	A
0	X	A	B

$A\ (15) >_{BC} B\ (14) >_{BC} C\ (13) >_{BC} X\ (0)$

$C\ (13) >_{BC} B\ (12) >_{BC} A\ (11) >_{BC} X\ (6)$