PHIL309P
Philosophy, Politics and Economics

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Collective decision making

Voter 1  >  > 

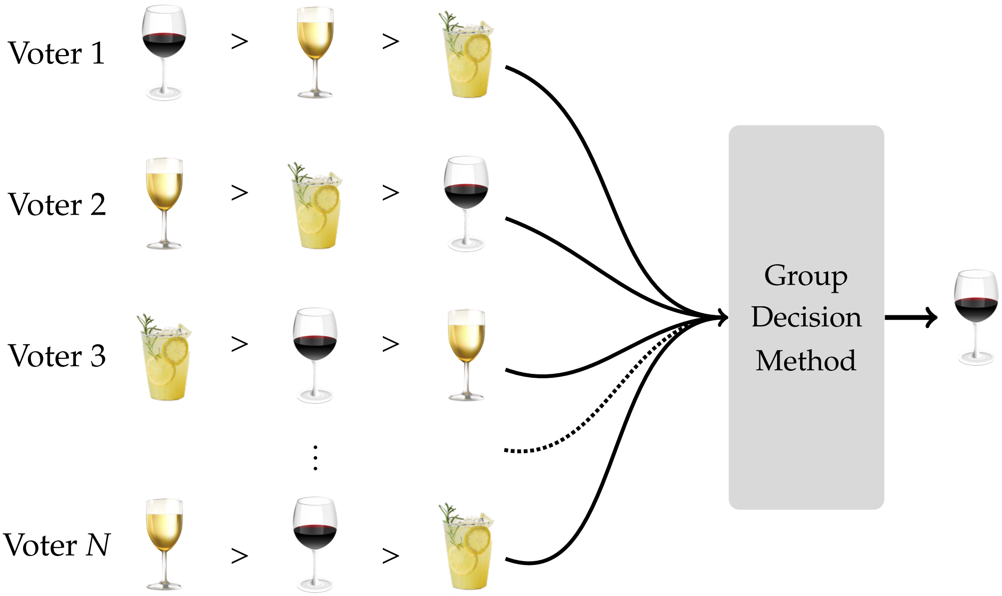
Voter 2  >  > 

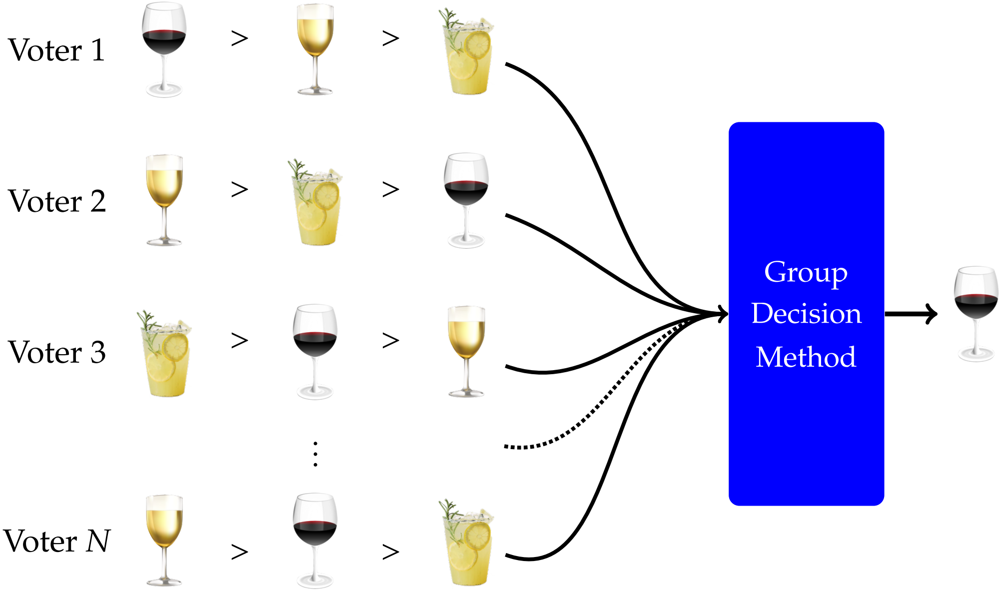
Voter 3  >  > 

$\vdots$

Voter N  >  > 


Voter 1
Voter 2
Voter 3
Voter N
>
Group Decision Method


Voter 1
Voter 2
Voter 3
Voter N
Group Decision Method


Voter 1
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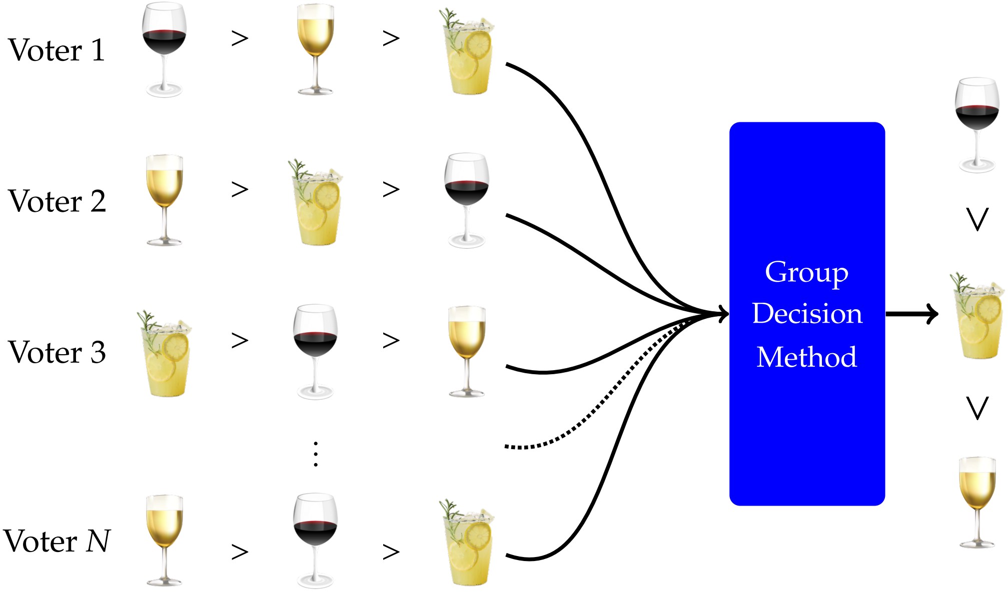
Group Decision Method


Voter 1
Voter 2
Voter 3
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Group Decision Method

Voting Situations

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- 21 voters and 4 candidates: Ann (A), Bob (B), Charles (C) and Dora (D)

Voting Situations

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best	A	A	B	C
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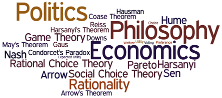
- 21 voters and 4 candidates: Ann (A), Bob (B), Charles (C) and Dora (D)
- Each voter ranks the candidates from best (at the top of the list) to worst (at the bottom of the list) resulting in the 4 voting blocks given in the above table

Voting Situations



# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

Who *should* win the election?

Which candidate *should* be chosen?

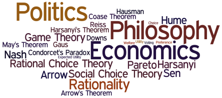
# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** *A*: More people (8) rank *A* first than any other candidate

Which candidate *should* be chosen?



# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	**A**	**A**

- **Candidate** *A*: More people rank *A* first than any other candidate
- **Candidate** *A* **should *not* win**: more than half rank *A* last

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** $\color{red}{A}$: More people rank A first than any other candidate
- **Candidate** D **should *not* win**

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

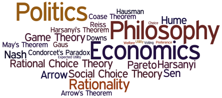
- **Candidate** *A*: More people rank *A* first than any other candidate
- **Candidate** *D* **should *not* win**: *everyone* ranks *B* higher than *D*

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Which of B or C should win?

Which candidate *should* be chosen?



VS.

Marquis de Condorcet (1743 - 1794)

Jean-Charles de Borda (1733 -1799)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win:** C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)

Which candidate *should* be chosen?

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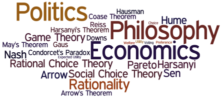

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best	A	A	B	C
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worst	D	D	A	A

- **Candidate** C **should win**: *C* beats every other candidate in head-to-head elections (*C* is the *Condorcet winner*)
- **Candidate** B **should win**: Taking into account the *entire* ordering, *B* has the most "support" (*B* is the *Borda winner*)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
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worst	D	D	A	A

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- **Candidate B should win**: Taking into account the *entire* ordering, *B* has the most "support" (*B* is the *Borda winner*)

- *B* gets 13 (vs. *A*)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
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worst	D	D	A	A

- **Candidate C should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- **Candidate B should win**: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)

- B gets 13 (vs. A) + 10 (vs. C)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- **Candidate B should win**: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)
- B gets 13 (vs. A) + 10 (vs. C) + 21 (vs. D) = 44 points

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- **Candidate B should win**: Taking into account the *entire* ordering, B has the most “support” (B is the *Borda winner*)

- C get 13 (vs. A) + 11 (vs. B) + 14 (vs. D) = 38 points

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
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Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate A should *not* win: more than half rank A last
- Candidate D should *not* win: *everyone* ranks B higher than D
- Candidate C: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- Candidate B: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)

Which candidate *should* be chosen?

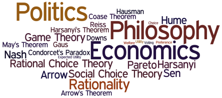
# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

Conclusion: there are many ways to answer the above question!

(C is the *Condorcet winner*)

- Candidate *B*: Taking into account the *entire* ordering, *B* has the most "support" (*B* is the *Borda winner*)

There are many different voting methods



Many different electoral methods: Plurality, Borda Count, Antiplurality/Veto, and k-approval; Plurality with Runoff; Single Transferable Vote (STV)/Hare; Approval Voting; Cup Rule/Voting Trees; Copeland; Banks; Slater Rule; Schwartz Rule; the Condorcet rule; Maximin/Simpson, Kemeny; Ranked Pairs/Tideman; Bucklin Method; Dodgson Method; Young's Method; Majority Judgment; Cumulative Voting; Range/Score Voting; . . .

Choosing how to choose

Pragmatic considerations: Is the procedure easy to use? Is it legal? The importance of ease of use should not be underestimated: Despite its many flaws, plurality rule is, by far, the most commonly used method.

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Information required from the voters: What type of information do the ballots convey? I.e., Choosing a single alternative, linearly rank all the candidates, report something about the "intensity" of preference.

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Information required from the voters: What type of information do the ballots convey? I.e., Choosing a single alternative, linearly rank all the candidates, report something about the "intensity" of preference.

Axiomatics: Characterize the different voting methods in terms of normative principles of group decision making.

Voting Methods

Positional Scoring Rules: Given the rankings of the candidates provided by the voters, each candidate is assigned a score. The candidate(s) with the highest score is(are) declared the winner(s).

Examples: Borda, Plurality

Generalized Scoring Rules: Voters assign scores, or "grades", to the candidates. The candidate(s) with the "best" aggregate score is(are) declared the winner(s).

Examples: Approval Voting, Majority Judgement, Range Voting

Voting Methods

Staged Procedures: The winner(s) is(are) determined in stages. At each stage, one or more candidates are eliminated. The candidate or candidates that are never eliminated are declared the winner(s).

Examples: Plurality with Runoff, Hare, Coombs

Condorcet Consistent Methods: Voting methods that guarantee that the Condorcet winner is elected.

Examples: Copeland, Dodgson, Young

Voting Methods Tutorial

The Condorcet Paradox

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	B	**C**
↑	B	**C**	D	B
	C	B	**C**	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections.

Recall Condorcet's Idea

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Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	**B**	C
	B	C	D	**B**
	C	**B**	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections. *B* is ranked second

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections. B is ranked second, D is ranked third, and A is ranked last.

$$C >_M B >_M D >_M A$$

The Majority Relation

Suppose that X and Y are candidates and P_i represents voter i's *strict preference*.

$\mathbf{N}(X\ P\ Y) = |\{i \mid X\ P_i\ Y\}|$
"the number of voters that rank X strictly above Y"

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"a majority prefers candidate X over candidate Y"

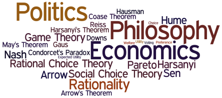
X is a **Condorcet winner** if X beats every other candidate in an head-to-head election: there is no candidate Y such that $Y >_M X$

X is a **Condorcet loser** if X loses to every other candidate in an head-to-head elections: there is no candidate Y such that, $X >_M Y$

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

The Problem



Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

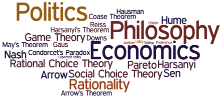
- Does the group prefer A over B?

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B? Yes

The Problem



Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B? Yes
- Does the group prefer B over C? Yes

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B? Yes
- Does the group prefer B over C? Yes
- Does the group prefer A over C? No

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

The majority relation $>_M$ is **not** transitive!

There is a **Condorcet cycle**: $A >_M B >_M C >_M A$

How bad is this?

- Final decisions are extremely sensitive to institutional features such as who can set the agenda, arbitrary time limits place on deliberation, who is permitted to make motions, etc.

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- Is there *empirical evidence* that Condorcet cycles have shown up in real elections?

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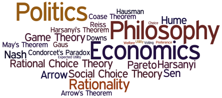
- How *likely* is a Condorcet cycle?

Should we select a Condorcet winner (when one exists)?

Condorcet's Other Paradox



# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
2	A	A	B	B	C	C
1	B	C	A	C	A	B
0	C	B	C	A	B	A

$$BS(A) = 2 \times 31 + 1 \times 39 + 0 \times 11 = 101$$
$$BS(B) = 2 \times 39 + 1 \times 31 + 0 \times 11 = 109$$
$$BS(C) = 2 \times 11 + 1 \times 11 + 0 \times 59 = 33$$

$$B >_{BC} A >_{BC} C$$

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# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

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Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$B >_{BC} A >_{BC} C$ $\quad\quad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
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Condorcet's Other Paradox: No *scoring rule* will work...

$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$

$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

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s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$

$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

$Score(A) > Score(B) \Rightarrow 31s_2 + 39s_1 > 39s_2 + 31s_1 \Rightarrow s_1 > s_2$

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Theorem (Fishburn 1974). For all $m \geq 3$, there is some voting situation with a Condorcet winner such that every scoring rule will have at least $m - 2$ candidates with a greater score than the Condorcet winner.

P. Fishburn. *Paradoxes of Voting*. The American Political Science Review, 68:2, pgs. 537 - 546, 1974.

# voters	30	1	29	10	10	1
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	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$$B >_{BC} A >_{BC} C \qquad\qquad A >_{M} B >_{M} C$$

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet Triples

G_1	G_2	G_3
A	B	C
B	C	A
C	A	B

G_1	G_2	G_3
A	C	B
C	B	A
B	A	C

If $G_1 = G_2 = G_3$, then this group of voters “cancel out” each other’s votes

Saari's argument

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

Saari's argument

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

10	10	10
A	B	C
B	C	A
C	A	B

Saari's argument

# voters	20	1	29	0	0	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

10	10	10
A	B	C
B	C	A
C	A	B

1	1	1
A	C	B
C	B	A
B	A	C

Saari's argument

# voters	20	0	28	0	0	0
	A		B			
	B		A			
	C		C			

10	10	10
A	B	C
B	C	A
C	A	B

1	1	1
A	C	B
C	B	A
B	A	C

Is the Condorcet winner the “best” choice?

# voters	47	47	3	3
	A	B	C	C
	C	C	A	B
	B	A	B	A

C is the Condorcet winner

Is the Condorcet winner the “best” choice?

# voters	47	47	3	3
	A	B	C	C
	C	C	A	B
	B	A	B	A

C is the Condorcet winner; however, it seems that supporters of the main rivals A and B would rather see C win than their candidate’s principal opponent, but this does not mean that there is “positive support” for C.

Approval Voting: Each voter selects a subset of candidates. The candidate with the most "approvals" wins the election.

S. Brams and P. Fishburn. *Approval Voting*. Birkhauser, 1983.

J.-F. Laslier and M. R. Sanver (eds.). *Handbook of Approval Voting*. Studies in Social Choice and Welfare, 2010.

Under Approval Voting (AV), voters are asked to select the candidates that the voter *approves*.

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Under ranking voting procedures (such as Borda Count), voters are asked to (linearly) rank the candidates.

The two pieces of information are related, but not derivable from each other

Approving of a candidate is not (necessarily) the same as simply ranking the candidate first.

Why Approval Voting?

www.electology.org/approval-voting

S. Brams and P. Fishburn. *Going from Theory to Practice: The Mixed Success of Approval Voting*. Handbook of Approval Voting, pgs. 19-37, 2010.

Approval Voting is more flexible

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is *A*.

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$, vote-for-2 elects $\{D\}$

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$, vote-for-2 elects $\{D\}$, vote-for-3 elects $\{A, B\}$.

Approval Voting is more flexible

AV may elect the Condorcet winner

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
$(\{A\}, \{B\}, \{C, A\})$ elects A under AV.

Possible Failure of Unanimity

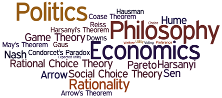
# voters	1	1	1
	A	C	D
	B	A	A
	C	B	B
	D	D	C

Possible Failure of Unanimity

# voters	1	1	1
	A	C	D
	B	A	A
	C	B	B
	D	D	C

Approval Winners: A, B

Indeterminate or Responsive?



# voters	6	5	4
	A	B	C
	C	C	B
	B	A	A

Plurality winner: *A*, Borda and Condorcet winner: *C*.

Indeterminate or Responsive?

# voters	6	5	4
	A	B	C
	C	C	B
	B	A	A

Plurality winner: *A*, Borda and Condorcet winner: *C*.
Any combination of *A*, *B* and *C* can be an AV winner (or AV winners).

Generalizing Approval Voting

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Ask the voters to provide both a linear ranking of the candidates and the candidates that they approve.

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Ask the voters to provide both a linear ranking of the candidates and the candidates that they approve.

Make the ballots more expressive: Dis&Approval voting, RangeVoting, Majority Judgement

Grading

In many group decision situations, people use measures or grades from a **common language of evaluation** to evaluate candidates or alternatives:

- in figure skating, diving and gymnastics competitions;
- in piano, flute and orchestra competitions;
- in classifying wines at wine competitions;
- in ranking university students;
- in classifying hotels and restaurants, e.g., the Michelin $*$

Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)

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Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)
- How should we *aggregate* the grades? (e.g., Average or Median)
- Should there be a "no opinion" option?

Voting by Grading: Examples

Approval Voting: voters can assign a single grade “approve” to the candidates

Dis&Approval Voting: voters can approve or disapprove of the candidates

Majority Judgement, Score Voting: voters can assign any grade from a fixed set of grades to the candidates

Strong Paradox of Grading Systems

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	$5/3$
B	0	3	1	$4/3$
C	0	3	1	$4/3$

Average Grade Winner: A

Grades: $\{0, 1, 2, 3\}$

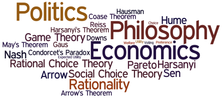
Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$B \succ A$$



Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$C \sim B \succ A$$

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$C \sim B \succ A$$

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

Superior Grade Winners: C, B

Grades: $\{0, 1, 2, 3, 4, 5\}$

Candidates: $\{A, B, C\}$

5 Voters

# voters	1	4	Avg
A	5	0	5/5
B	0	1	4/5
C	0	1	4/5

Average Grade Winner: A

Superior Grade Winner: B, C

To conclude, we have identified a paradox of grading systems, which is not just a mirror of the well-known differences that crop up in aggregating votes under ranking systems. Unlike these systems, for which there is no accepted way of reconciling which candidate to choose when, for example, the Hare, Borda and Condorcet winners differ, AV provides a solution when the AG and SG winners differ.

Theorem (Brams and Potthoff). When there are two grades, the AG and SG winners are identical.

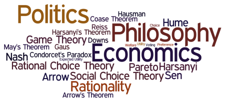
Further Investigation

- W. Poundstone, *Gaming the Vote: Why Elections Aren't Fair (and What We Can Do About It)*, Hill and Wang, 2009
- EP, Voting Methods (Stanford Encyclopedia of Philosophy)
- C. List, Social Choice Theory (Stanford Encyclopedia of Philosophy)
- M. Morreau, Arrow's Theorem (Stanford Encyclopedia of Philosophy)

Further Investigation

- https://www.electology.org
- http://www.fairvote.org
- http://rangevoting.org
- https://www.opavote.com
- http://www.preflib.org

There are many different voting methods



Many different electoral methods: Plurality, Borda Count, Antiplurality/Veto, and k-approval; Plurality with Runoff; Single Transferable Vote (STV)/Hare; Approval Voting; Cup Rule/Voting Trees; Copeland; Banks; Slater Rule; Schwartz Rule; the Condorcet rule; Maximin/Simpson, Kemeny; Ranked Pairs/Tideman; Bucklin Method; Dodgson Method; Young's Method; Majority Judgment; Cumulative Voting; Range/Score Voting; ...

Choosing how to choose

Pragmatic considerations: Is the procedure easy to use? Is it legal? The importance of ease of use should not be underestimated: Despite its many flaws, plurality rule is, by far, the most commonly used method.

Behavioral considerations: Do the different procedures *really* lead to different outcomes in practice?

Information required from the voters: What type of information do the ballots convey? I.e., Choosing a single alternative, linearly rank all the candidates, report something about the "intensity" of preference.

Axiomatics: Characterize the different voting methods in terms of normative principles of group decision making.

Principles of group decision making

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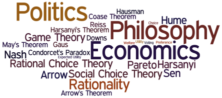
- **Condorcet Condition**: Always choose the candidate that beats every other candidate in head-to-head elections.

Principles of group decision making



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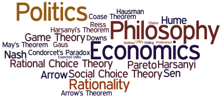
Principles of group decision making



- **Condorcet Condition**: Always choose the candidate that beats every other candidate in head-to-head elections.

- **Unanimity (Pareto)**: If *everyone* ranks A above B, then B should not win the election.

- **Anonymity**: The names of the voters do not matter (if two voters swap votes, then the outcome is unaffected).

Monotonicity

A candidate receiving more “support” shouldn’t maker her worse off.

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A candidate receiving more "support" shouldn't maker her worse off.

More-is-Less Paradox: If a candidate C is elected under a given a profile of rankings of the competing candidates, it is possible that, *ceteris paribus*, C may not be elected if some voter(s) raise C in their rankings.

P. Fishburn and S. Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine (1983).

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

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# voters	6	5	4	2
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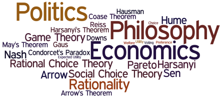

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

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	A	C	B	B
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	A	C	B	A
	B	A	C	B
	C	B	A	C

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: A

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: *A*

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: *A*

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Winner: *C*

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	**B**
	B	A	C	**A**
	C	B	A	**C**

Winner: *A*

# voters	6	5	4	2
	A	C	B	**A**
	B	A	C	**B**
	C	B	A	**C**

Winner: *C*

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- **Twin Paradox**: A voter may obtain a less preferable outcome if his "twin" (a voter with the exact same ranking) decides to participate in the election.

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- **Twin Paradox**: A voter may obtain a less preferable outcome if his "twin" (a voter with the exact same ranking) decides to participate in the election.
- **Truncation Paradox**: A voter may obtain a more preferable outcome if, *ceteris paribus*, he only reveals part of his ranking of the candidates.

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: C

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: C

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: C

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: *C*

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: *B*

Twin Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: *C*

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: *B*

Failures of Monotonicity

Example: Burlington, VT 2009 Mayoral Race
(rangevoting.org/Burlington.html)

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D. Felsenthal and N. Tideman. *Varieties of Failure of Monotonicity and Participation under Five Voting Methods*. Theory and Decision, 75, pgs. 59 - 77, 2013.

Failures of Monotonicity

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Theorem (Moulin). If there are four or more candidates, then every Condorcet consistent voting methods is susceptible to the No-Show paradox.

H. Moulin. *Condorcet's Principle Implies the No Show Paradox*. Journal of Economic Theory, 45, pgs. 53 - 64, 1988.

Spoiler Candidates: Plurality Rule

# voters	49	48	3
	A	B	C
	B	A	B
	C	C	A

Winner: *A*

Spoiler Candidates: Plurality Rule

# voters	49	48	3
	A	B	C
	B	A	B
	C	C	A

Winner: *B*

IIA

Independence of Irrelevant Alternatives: If the voters in two different electorates rank A and B in exactly the same way, then A and B should be ranked the same way in both elections.

Failure of IIA: Borda Count

# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

# voters	3	2	2
	A	B	C
	B	C	X
	C	X	A
	X	A	B

Failure of IIA: Borda Count

# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

A (15) $>_{BC}$ B (14) $>_{BC}$ C (13) $>_{BC}$ X (0)

# voters	3	2	2
	A	B	C
	B	C	X
	C	X	A
	X	A	B

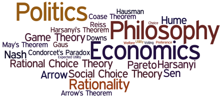
Failure of IIA: Borda Count

# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

# voters	3	2	2
3	A	B	C
2	B	C	X
1	C	X	A
0	X	A	B

$A\ (15) >_{BC} B\ (14) >_{BC} C\ (13) >_{BC} X\ (0)$

Failure of IIA: Borda Count



# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

$A\ (15) >_{BC} B\ (14) >_{BC} C\ (13) >_{BC} X\ (0)$

# voters	3	2	2
3	A	B	C
2	B	C	X
1	C	X	A
0	X	A	B

$C\ (13) >_{BC} B\ (12) >_{BC} A\ (11) >_{BC} X\ (6)$

Principles

Condorcet: Elect the Condorcet winner whenever it exists.

Monotonicity: More support should never hurt a candidate.

Participation: It should never be in a voter's best interests not to vote.

Multiple-Districts: If a candidate wins in each district, then that candidate should also win when the districts are merged.

Independence: The group's ranking of A and B should only depend on the voter's rankings of A and B.

More Principles

Pareto: Never elect a candidate if another candidate is strictly preferred by all voters.

Anonymity: The outcome does not depend on the names of the voters.

Neutrality: The outcome does not depend on the names of the candidates.

Universal Domain: The voters are free to rank the candidates (or grade the candidates) in any way they want.

What are the relationships between these principles? Is there a procedure that satisfies *all* of them?

What are the relationships between these principles? Is there a procedure that satisfies *all* of them?

A few observations:

- Condorcet winners may not exist.
- No positional scoring method satisfies the Condorcet Principle.
- The Condorcet and Participation principles cannot be jointly satisfied.

Axiomatics

"When a set of axioms regarding social choice can all be simultaneously satisfied, there may be several possible procedures that work, among which we have to choose.

A. Sen. *The Possibility of Social Choice*. The American Economic Review, 89:3, pgs. 349 - 378, 1999 (reprint of his Nobel lecture).

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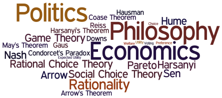
Axiomatics

"When a set of axioms regarding social choice can all be simultaneously satisfied, there may be several possible procedures that work, among which we have to choose. In order to choose between different possibilities through the use of discriminating axioms, we have to introduce *further* axioms, until only and only one possible procedure remains. This is something of an exercise in brinkmanship. We have to go on and on cutting alternative possibilities, moving—implicitly—*towards* an impossibility, but then stop just before all possibilities are eliminated, to wit, when one and only one options remains." (pg. 354)

A. Sen. *The Possibility of Social Choice.* The American Economic Review, 89:3, pgs. 349 - 378, 1999 (reprint of his Nobel lecture).

The Social Choice Model

Notation

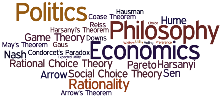


- N is a finite set of voters (assume that $N = \{1, 2, 3, \ldots, n\}$)
- X is a (typically finite) set of alternatives, or candidates
- A relation on X is a linear order if it is transitive, irreflexive, and complete (hence, acyclic)
- $L(X)$ is the set of all linear orders over the set X
- $O(X)$ is the set of all reflexive and transitive relations over the set X

Notation

- A **profile** for the set of voters N is a sequence of (linear) orders over X, denoted $\mathbf{R} = (R_1, \ldots, R_n)$.
- $L(X)^n$ is the set of all **profiles** for n voters (similarly for $O(X)^n$)
- For a profile $\mathbf{R} = (R_1, \ldots, R_n) \in O(X)^n$, let $\mathbf{N_R}(A\ P\ B) = \{i \mid A\ P_i\ B\}$ be the set of voters that rank A above B (similarly for $\mathbf{N_R}(A\ I\ B)$ and $\mathbf{N_R}(B\ P\ A)$)

Preference Aggregation Methods



Social Welfare Function: $F : \mathcal{D} \to L(X)$, where $\mathcal{D} \subseteq L(X)^n$

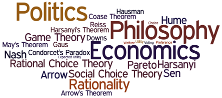
Preference Aggregation Methods

Social Welfare Function: $F : \mathcal{D} \rightarrow L(X)$, where $\mathcal{D} \subseteq L(X)^n$

Comments

- $\mathcal{D}$ is the *domain* of the function: it is the set of all possible profiles
- Aggregation methods are *decisive*: every profile $\mathbf{R}$ in the domain is associated with exactly one ordering over the candidates
- The range of the function is $L(X)$: the social ordering is assumed to be a linear order
- Tie-breaking rules are built into the definition of a preference aggregation function

Preference Aggregation Methods



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Variants

- Social Choice Function: $F : \mathcal{D} \rightarrow \wp(X) - \emptyset$, where $\mathcal{D} \subseteq L(X)^n$ and $\wp(X)$ is the set of all subsets of X.

- Allow Ties: $F : \mathcal{D} \rightarrow O(X)$ where $O(X)$ is the set of orderings (reflexive and transitive) over X

- Allow Indifference and Ties: $F : \mathcal{D} \rightarrow O(X)$ where $O(X)$ is the set of orderings (reflexive and transitive) over X and $\mathcal{D} \subseteq O(X)^n$

Examples

$Maj(\mathbf{R}) = \ >_M$ where $A >_M B$ iff $|\mathbf{N}_{\mathbf{R}}(A\ P\ B)| > |\mathbf{N}_{\mathbf{R}}(B\ P\ A)|$

(*the problem is that* $>_M$ *may not be transitive (or complete)*)

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$Borda(\mathbf{R}) \ = \ \geq_{BC}$ where $A \geq_{BC} B$ iff the Borda score of A is greater than the Borda score for B.

(*the problem is that* $\geq_{BC}$ *may not be a linear order*)

Characterizing Majority Rule

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When there are only two options, can we argue that majority rule is the "best" procedure?

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

May's Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of n voters and $X = \{A, B\}$ the set of candidates.

Social Welfare Function: $F : O(X)^n \to O(X)$, where $O(X)$ is the set of orderings over X
(*there are only three possibilities:* $A\ P\ B$, $A\ I\ B$, *or* $B\ P\ A$)

$$F_{Maj}(\mathbf{R}) = \begin{cases} A\ P\ B & \text{if } |\mathbf{N_R}(A\ P\ B)| > |\mathbf{N_R}(B\ P\ A)| \\ A\ I\ B & \text{if } |\mathbf{N_R}(A\ P\ B)| = |\mathbf{N_R}(B\ P\ A)| \\ B\ P\ A & \text{if } |\mathbf{N_R}(B\ P\ A)| > |\mathbf{N_R}(A\ P\ B)| \end{cases}$$

May's Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of n voters and $X = \{A, B\}$ the set of candidates.

Social Welfare Function: $F : \{1, 0, -1\}^n \rightarrow \{1, 0, -1\}$,

where 1 means $A\ P\ B$, 0 means $A\ I\ B$, and -1 means $B\ P\ A$

$$F_{Maj}(\mathbf{v}) = \begin{cases} 1 & \text{if } |\mathbf{N_v}(1)| > |\mathbf{N_v}(-1)| \\ 0 & \text{if } |\mathbf{N_v}(1)| = |\mathbf{N_v}(-1)| \\ -1 & \text{if } |\mathbf{N_v}(-1)| > |\mathbf{N_v}(1)| \end{cases}$$

Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there?

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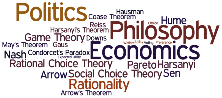

Suppose that there are two voters and two candidates. How many social choice functions are there? $19,683$

- There are three possible rankings for 2 candidates.
- When there are two voters there are $3^2 = 9$ possible profiles:

$$\{(1,1),(1,0),(1,-1),(0,1),(0,0),(0,-1),(-1,1),(-1,0),(-1,-1)\}$$

- Since there are 9 profiles and 3 rankings, there are $3^9 = 19,683$ possible preference aggregation functions.

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.

- **Anonymity**: all voters should be treated equally.

- **Neutrality**: all candidates should be treated equally.

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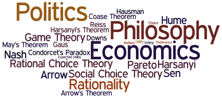

- **Unanimity**: unanimously supported alternatives must be the social outcome.

  If $\mathbf{v} = (v_1, \ldots, v_n)$ with for all $i \in N$, $v_i = x$ then $F(\mathbf{v}) = x$ (for $x \in \{1, 0, -1\}$).

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  $F(v_1, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where $v_i \in \{1, 0, -1\}$ and π is a permutation of the voters.

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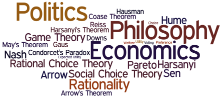

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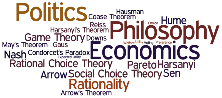
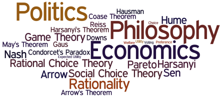
  $F(-\mathbf{v}) = -F(\mathbf{v})$ where $-\mathbf{v} = (-v_1, \ldots, -v_n)$.

May’s Theorem: Details



- **Positive Responsiveness** (Monotonicity): unidirectional shift in the voters’ opinions should help the alternative toward which this shift occurs

> If $F(\mathbf{v}) = 0$ or $F(\mathbf{v}) = 1$ and $\mathbf{v} \prec \mathbf{v}'$, then $F(\mathbf{v}') = 1$
> where $\mathbf{v} \prec \mathbf{v}'$ means for all $i \in N$ $v_i \leq v_i'$ and there is some $i \in N$ with $v_i < v_i'$.



Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there that satisfy anonymity?

Anonymity: all voters should be treated equally.

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Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there that satisfy anonymity? 729

Anonymity: all voters should be treated equally.

$F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

- Imposing anonymity reduces the number of preference aggregation functions.
- If F satisfies anonymity, then $F(1,0) = F(0,1)$, $F(1,-1) = F(-1,1)$ and $F(-1,0) = F(0,-1)$.
- This means that there are essentially 6 elements of the domain. So, there are $3^6 = 729$ preference aggregation functions.

May's Theorem: Details

May's Theorem (1952) A social decision method F satisfies unanimity, neutrality, anonymity and positive responsiveness iff F is majority rule.

Proof Idea

If $(1, 0, -1)$ is assigned 1 or -1 then

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✓ Neutrality implies $(1, 0, -1)$ is assigned -1 or 1
Contradiction.

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  Contradiction.

Other characterizations

G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

Can May's Theorem be generalized to more than 2 candidates?

Can May's Theorem be generalized to more than 2 candidates? No!