PHIL309P
Methods in Philosophy, Politics and Economics

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Strategic Games

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- N is a finite set of **players**

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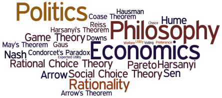
- N is a finite set of **players**
- for each $i \in N$, A_i is a nonempty set of **actions**
- for each $i \in N$, $\succeq_i$ is a **preference relation** on $A = \Pi_{i \in N} A_i$
  (Often $\succeq_i$ are represented by **utility functions** $u_i : A \to \mathbb{R}$)

Strategic Games: Comments on Preferences



- Preferences may be over a set of consequences C. Assume $g : A \to C$ and $\{\succeq_i^* \mid i \in N\}$ a set of preferences on C. Then for $a, b \in A$,

$$a \succeq_i b \text{ iff } g(a) \succeq_i^* g(b)$$

- Consequences may be affected by exogenous random variable whose realization is not known before choosing actions. Let Ω be a set of states, then define $g : A \times \Omega \to C$. Where $g(a|\cdot)$ is interpreted as a *lottery*.

- Often $\succeq_i$ are represented by **utility functions** $u_i : A \to \mathbb{R}$

Strategic Games: Example

		Column	
		r	l
Row	u		
	d		

- $N = \{Row, Column\}$
- $A_{Row} = \{u, d\}, A_{Column} = \{r, l\}$
- $(u, r) \succeq_{Row} (d, l) \succeq_{Row} (u, l) \sim_{Row} (d, r)$
  $(u, r) \succeq_{Column} (d, l) \succeq_{Column} (u, l) \sim_{Column} (d, r)$

Strategic Games: Example

Row		Column	
		r	l
	u	(2,2)	(0,0)
	d	(0,0)	(1,1)

- $N = \{Row, Column\}$
- $A_{Row} = \{u, d\}$, $A_{Column} = \{r, l\}$
- $u_{Row} : A_{Row} \times A_{Column} \to \{0, 1, 2\}$, $u_{Column} : A_{Row} \times A_{Column} \to \{0, 1, 2\}$ with $u_{Row}(u, r) = u_{Column}(u, r) = 2$, $u_{Row}(d, l) = u_{Column}(d, l) = 2$, and $u_x(u, l) = u_x(d, r) = 0$ for $x \in N$.

Some Types of Games

Cooperative/non-cooperative: in *all* types of strategic games we assume that players are self-interested (i.e., utility maximizers).

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Pure Coordination

		Bob	
		L	R
Ann	U	1,1	0,0
	D	0,0	1,1

Focal Points

'primary salience': players' psychological propensities to play particular strategies by default, when there are no other reasons for choice.

"The basic intellectual premise, or working hypothesis, for rational players in this game seems to be the premise that some rule must be used if success is to exceed coincidence, and that the best rule to be found, whatever its rationalization, is consequently a rational rule." (pg. 283)

T. Schelling. *The Strategy of Conflict*. Harvard University Press.

Hi-Low

Ann \ Bob	L	R
U	3,3	0,0
D	0,0	1,1

Pareto Dominant/Focal Points

> "There are these two broad empirical facts about Hi-Lo games, people almost always choose *A* [Hi] and people with common knowledge of each other's rationality think it is obviously rational to choose *A* [Hi]." (pg. 42)

M. Bacharach. *Beyond Individual Choice*. Princeton University Press, 2006.

See also chapter 2 of:

C.F. Camerer. *Behavioral Game Theory*. Princeton Princeton University Press, 2003.

Zero-sum/nonzero-sum: zero-sum games describe situations where there is a fixed amount of "goods" (i.e., utility) to be distributed amongst the players, so one player getting more means that the remaining players get less.

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Suppose there are two players Ann and Bob dividing a cake. Suppose that Ann cuts the cake and then Bob chooses the first piece. (Suppose they *only* care about the size of the piece). Ann cannot cut the cake exactly evenly, so one piece is always larger than the other.

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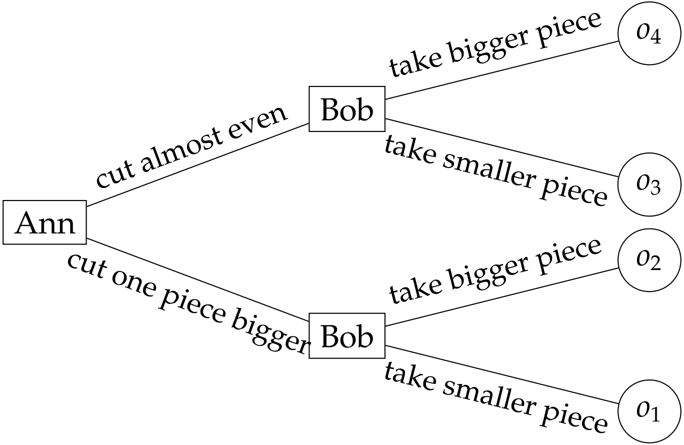

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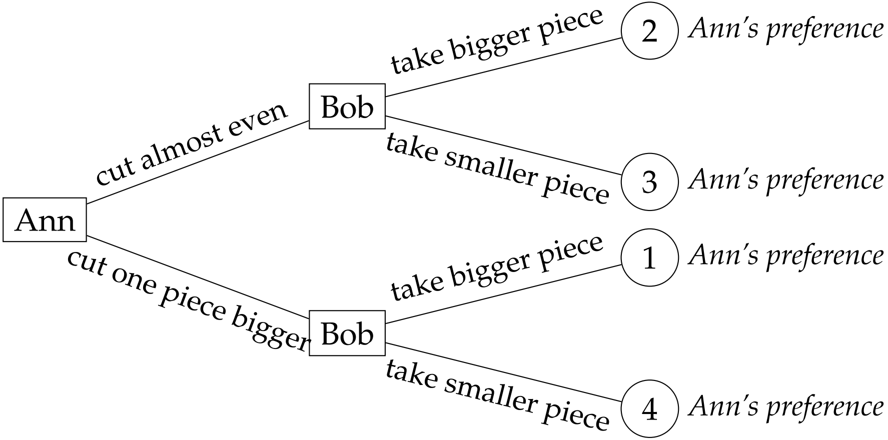

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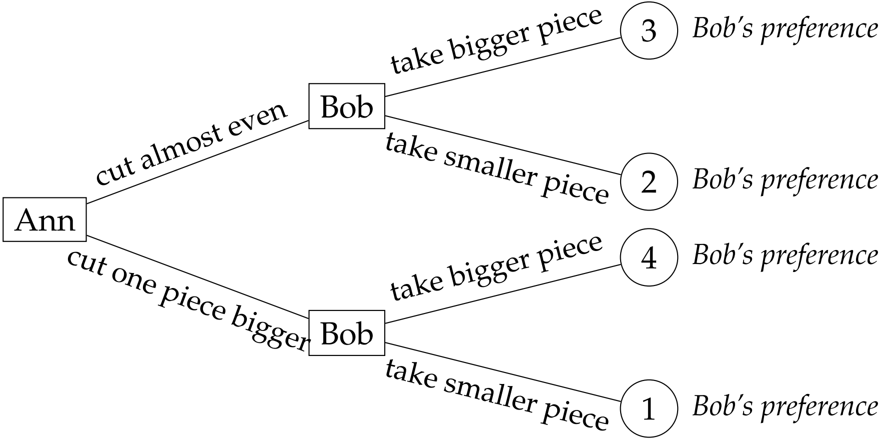
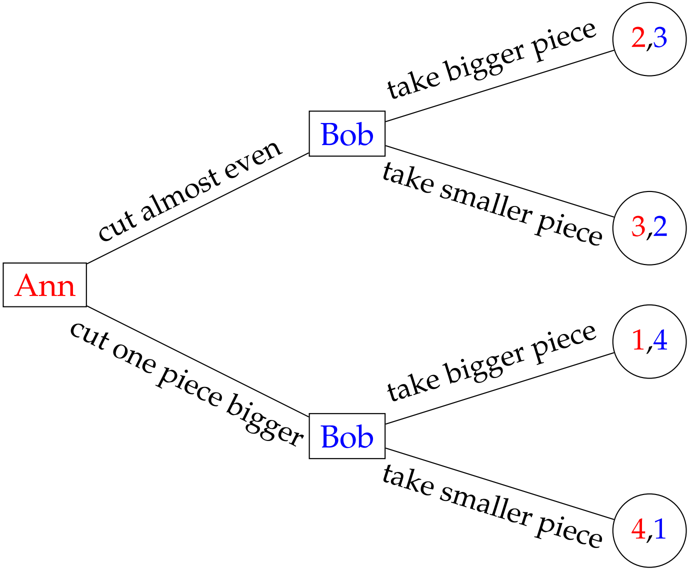



Ann
cut almost even
cut one piece bigger
Bob
take bigger piece
take smaller piece
2,3
3,2
Bob
take bigger piece
take smaller piece
1,4
4,1

		Bob	
		TB	TS
Ann	CB	1,4	4,1
	CE	2,3	3,2

What should Ann (or Bob) *do*?

		Bob	
		TB	*TS*
Ann	*CB*	1,4	4,1
	CE	2,3	3,2

What should Ann (or Bob) *do*? Ann's best choice in Bob's worst choice (and vice versa)

In *zero-sum* games it is as if players explicitly want to minimize the pay-off of others, which is *not* true of games in general

		Bob	
		TB	TS
Ann	CB	1,4	4,1
	CE	2,3	3,2

What should Ann *do*?

		Bob	
		TB	*TS*
Ann	*CB*	1,4	4,1
	CE	2,3	3,2

What should Ann *do*? *Bob best choice in Ann's worst choice*

	Bob		
Ann	*TB*	*TS*	
CB	1,4	4,1	1
CE	2,3	3,2	2

What should Ann *do*? *maximize over each row and choose the maximum value*

		Bob	
		TB	*TS*
Ann	*CB*	1,4	4,1
	CE	2,3	3,2
		3	1

What should Bob *do*? *minimize over each column and choose the maximum value*

Zero-Sum Games

Von Neumann Minmax Theorem. In any finite, two-player, zero-sum game, there is always at least one minmax solution.

Matching Pennies

		Bob	
		H	T
Ann	H	1,-1	-1, 1
	T	-1,1	1,-1

Matching Pennies

		Bob	
		H	T
Ann	H	1,-1	-1, 1
	T	-1,1	1,-1

There are no equilibrium.

Mixed Strategies

		Bob	
		H	T
Ann	H	1,-1	-1, 1
	T	-1,1	1,-1

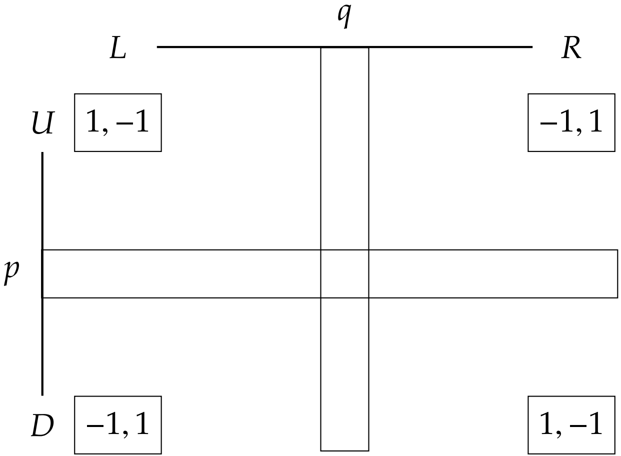
A **mixed strategy** is a probability distribution over the set of pure strategies. For instance:

- $[1/2 : H, 1/2 : T]$
- $[1/3 : H, 2/3 : T]$
- ...

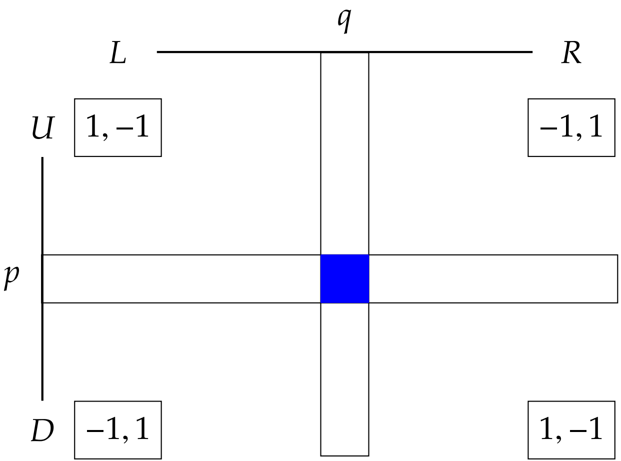
Mixed Extension

	L	R
U	$1, -1$	$-1, 1$
D	$-1, 1$	$1, -1$

Mixed Extension



Mixed Extension



$$pq - p(1-q) - (1-p)q + (1-p)(1-q),\ -pq + p(1-q) + (1-p)q - (1-p)(1-q)$$

Matching Pennies

Ann (rows) vs. Bob (columns)

	H	T
H	1,-1	-1, 1
T	-1,1	1,-1

The mixed strategy $([1/2 : H, 1/2 : T], [1/2 : H, 1/2 : T])$ is the only equilibrium.

Theorem (von Neumann). For every two-player zero- sum game with finite strategy sets S_1 and S_2, there is a number v, called the **value** of the game such that:

1. $v = \max_{p\in\Delta(S_1)} \min_{q\in\Delta(S_2)} U_1(p,q) = \min_{q\in\Delta(S_2)} \max_{p\in\Delta(S_1)} U_1(p,q)$
2. The set of mixed equilibria is nonempty. A mixed strategy profile (p,q) is a Nash equilibrium if and only if

$$p \in \operatorname{argmax}_{p\in\Delta(S_1)} \min_{q\in\Delta(S_2)} U_1(p,q)$$

$$q \in \operatorname{argmax}_{q\in\Delta(S_2)} \min_{p\in\Delta(S_1)} U_1(p,q)$$

3. For all mixed equilibria (p,q), $U_1(p,q) = v$

		Bob	
		B	S
Ann	B	2, 1	0, 0
	S	0, 0	1, 2

Nash Equilibrium

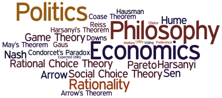
Let $\langle N, \{A_i\}_{i \in N}, \{\succeq_i\}_{i \in N} \rangle$ be a strategic game

For $a_{-i} \in A_{-i}$, let

$$B_i(a_{-i}) = \{a_i \in A_i \mid (a_{-i}, a_i) \succeq_i (a_{-i}, a_i') \;\forall\, a_i' \in A_i\}$$

B_i is the **best-response** function.

Nash Equilibrium



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B_i is the **best-response** function.

$a^* \in A$ is a **Nash equilibrium** iff $a_i^* \in B_i(a_{-i}^*)$ for all $i \in N$.

Example

	L	R
U	2,1	0,0
D	0,0	1,2

Example

	L	R
U	2,1	0,0
D	0,0	1,2

$$N = \{r, c\} \quad A_r = \{U, D\} \quad A_c = \{L, R\}$$

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	L	R
U	2,1	0,0
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$$N = \{r, c\} \quad A_r = \{U, D\} \quad A_c = \{L, R\}$$

$$BR_r(L) = \{U\} \qquad\qquad BR_r(R) = \{D\}$$

Example

	L	R
U	2,1	0,0
D	0,0	1,2

$$N = \{r, c\} \quad A_r = \{U, D\} \quad A_c = \{L, R\}$$

$$BR_r(L) = \{U\} \qquad BR_r(R) = \{D\}$$

$$BR_c(U) = \{L\} \qquad BR_c(D) = \{R\}$$

Example

	L	R
U	2,1	0,0
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$$N = \{r, c\} \quad A_r = \{U, D\} \quad A_c = \{L, R\}$$

$$BR_r(L) = \{U\} \qquad BR_r(R) = \{D\}$$

$$BR_c(U) = \{L\} \qquad BR_c(D) = \{R\}$$

(U, L) is a Nash Equilibrium $\qquad$ (D, R) is a Nash Equilibrium

Zero-Sum Games

		Bob	
		L	R
Ann	U	1,4	4,1
	D	2,3	3,2

The profile of security strategies (D, L) is a Nash equilbirium

In zero-sum games

- There exists a mixed strategy Nash equilibrium
- There may be more than one Nash equilibria
- Security strategies are always a Nash equilibrium
- Components of Nash equilibria are interchangeable: If σ and σ' are Nash equilibria in a 2-player game, then (σ_1, σ_2') is also a Nash equilibrium.

Battle of the Sexes

		Bob	
		B	S
Ann	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

Battle of the Sexes

		Bob	
		B	S
Ann	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

(B, B) (S, S), and $([2/3 : B, 1/3 : S], [1/3 : B, 2/3 : S])$ are Nash equilibria.

In an arbitrary (finite) games (that are not zero-sum)

- There exists a mixed strategy Nash equilibrium
- Security strategies are not necessarily a Nash equilibrium
- There may be more than on Nash equilibrium
- Components of Nash equilibrium are not interchangeable.
- Why *should* players play a Nash equilibrium?

Let $G = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a finite strategic game (each S_i is finite and the set of players N is finite).

A **strategy profile** is an element $\sigma \in S = S_1 \times \cdots \times S_n$

σ is a **Nash equilibrium** provided for all i, for all $s_i \in S_i$,

$$u_i(\sigma) \geq u_i(s_i, \sigma_{-i})$$