PHIL309P
Methods in Philosophy, Politics and Economics

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Ordinal Utility Theory

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Fact. Suppose that X is finite and $\succeq$ is a complete and transitive ordering over X, then there is a utility function $u : X \to \mathfrak{R}$ that represents $\succeq$

(i.e., $x \succeq y$ iff $u(x) \geq u(y)$)

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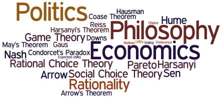

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(i.e., $x \succeq y$ iff $u(x) \geq u(y)$)

Utility is *defined* in terms of preference (so it is an error to say that the agent prefers x to y *because* she assigns a higher utility to x than to y).

Important

All three of the utility functions represent the preference $x \succ y \succ z$

Item	u_1	u_2	u_3
x	3	10	1000
y	2	5	99
z	1	0	1

$x \succ y \succ z$ is represented by both $(3, 2, 1)$ and $(1000, 999, 1)$, so one cannot say that y is "closer" to x than to z.

Lotteries

Suppose that X is a set of outcomes.

A **(simple) lottery** over X is denoted $[x_1 : p_1, x_2 : p_2, \ldots, x_n : p_n]$ where for $i = 1, \ldots, n$, $x_i \in X$ and $p_i \in [0, 1]$, and $\sum_i p_i = 1$.

Let $\mathcal{L}$ be the set of (simple) lotteries over X. We identify elements $x \in X$ with the lottery $[x : 1]$.

Lotteries

Suppose that $X = \{x_1, \ldots, x_n\}$ is a set of outcomes. A **lottery** over X is a tuple $[x_1 : p_1, x_2 : p_2, \ldots, x_n : p_n]$ where $\sum_i p_i = 1$.

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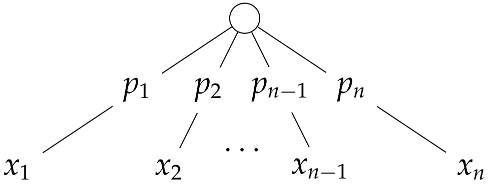

Let $\mathcal{L}$ be the set of lotteries.

Expected monetary value

Suppose that the outcomes of a lottery are monetary values. So, $L = [x_1 : p_1, x_2 : p_2, \ldots, x_n : p_n]$, where each x_i is an amount of money. Then,

$$EV(L) = \sum_i p_i \times x_i$$

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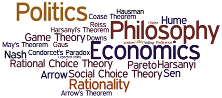

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E.g., if $L = [\$100 : 0.55, \$50 : 0.25, \$0 : 0.20]$, then

$$EV(L) = 0.55 * 100 + 0.25 * 50 + 0.2 * 0 = 80$$

Problems with using monetary payoffs

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- The St. Petersburg Paradox: Consider the following wager: I will flip a fair coin until it comes up heads; if the first time it comes up heads is the n^{th} toss, then I will pay you 2^n. What's the most you'd be willing to pay for this wager? What is its expected monetary value?

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- Valuing Money: Doesn't the value of a wager depend on more than merely how much it's expected to pay out? (I.e., your total fortune, how much you personally care about money, etc.)
- Risk-aversion: Is it irrational to prefer a sure-thing $x to a wager whose expected payout is $x?

We should move away from “monetary payouts” to “utility”.

Expected Utility

Suppose that $X = \{x_1, \ldots, x_n\}$ and $u : X \to \mathbb{R}$ is a utility function on X.

This can be extended to an expected utility function $EU : \mathcal{L}(X) \to \mathbb{R}$ where

$$\begin{aligned} EU([x_1 : p_1, \ldots, x_n : p_n]) &= p_1 \times u(x_1) + \cdots + p_n \times u(x_n) \\ &= \sum_{i=1}^{n} p_i \times u(x_i) \end{aligned}$$

Suppose that Ann is faced with the choice between lotteries L_1 and L_2 where:

$$L_1 = [o_1 : 0,\ o_3 : 0.25,\ o_3 : 0.75] \qquad\qquad L_2 = [o_1 : 0.2,\ o_2 : 0,\ o_3 : 0.8]$$

Can expected utility theory tell us how Ann should rank L_1 and L_2?

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Suppose that Ann is also faced with the choice between lotteries L_3 and L_4 where:

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If we know that Ann ranks L_1 over L_2 (e.g., $L_1 \succ L_2$), can we conclude anything about how Ann ranks L_3 and L_4? Yes: Ann must rank L_4 over L_3 (e.g., $L_4 \succ L_3$).

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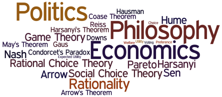
Which comparisons are meaningful?

1. $u(x)$ and $u(y)$? (ordinal utility)
2. $u(x) - u(y)$ and $u(a) - u(b)$?
3. $u(x)$ and $2 * u(z)$?

Ordinal vs. Cardinal Utility

Ordinal scale: Qualitative comparisons of objects allowed, no information about differences or ratios.

Ordinal vs. Cardinal Utility



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Cardinal scales:

> **Interval scale**: Quantitative comparisons of objects, accurately reflects differences between objects.
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> **Ratio scale**: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10lb (= 4.53592kg) is twice as much as 5lb (= 2.26796kg).

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Key idea: Ordinal preferences over *lotteries* allows us to infer a cardinal (interval) scale (with some additional axioms).

John von Neumann and Oskar Morgenstern. *The Theory of Games and Economic Behavior*. Princeton University Press, 1944.

A Choice

R

B

W

S

A Choice

R

B

W

S

Take or Gamble?

Take — B

Gamble

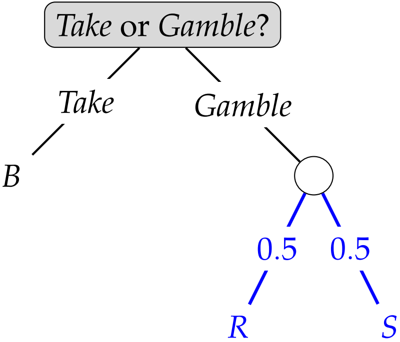
A Choice

R

B

W

S



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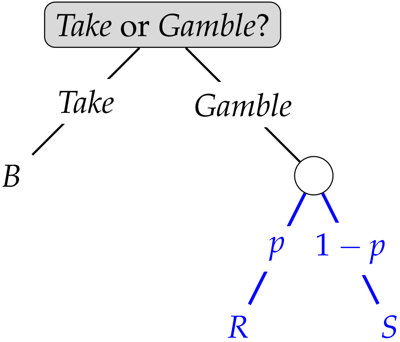

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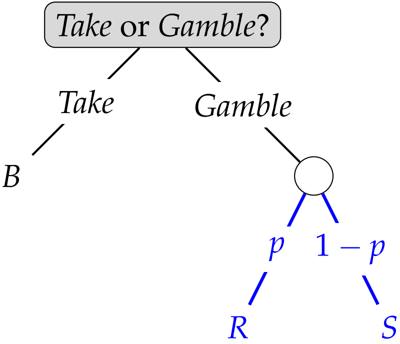

$$[1 : B] \sim [p : R, 1 - p : S]$$

A Choice

R

B

W

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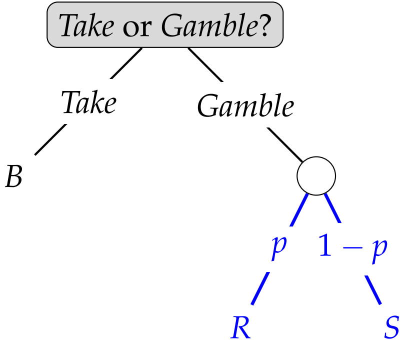

$$1 * u(B) = p * u(R) + (1 - p) * u(S)$$

A Choice

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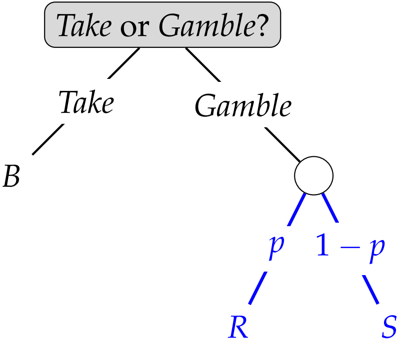
B

W

S



$$u(B) = p * 1 + (1 - p) * 0 = p$$

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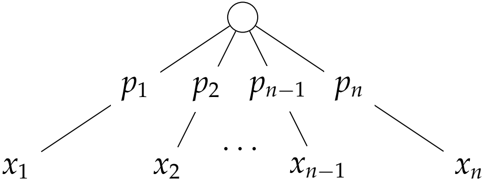

Let $\mathcal{L}$ be the set of lotteries. Suppose that $\succeq \subseteq \mathcal{L} \times \mathcal{L}$ is a preference ordering on $\mathcal{L}$.

Reduction of Compound Lotteries

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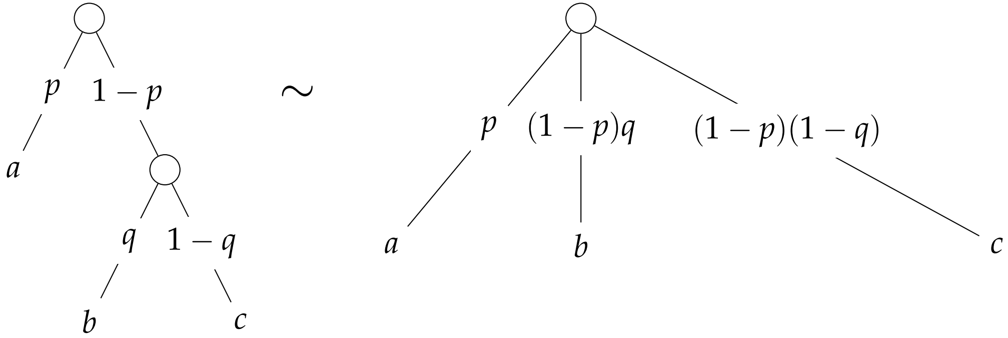
This eliminates utility from the thrill of gambling and so the only ultimate concern is the prizes.


p
1 − p
a
q
1 − q
b
c
∼
p
(1 − p)q
(1 − p)(1 − q)
a
b
c

Continuity

Continuity For all $L_1, L_2, L_3 \in \mathcal{L}$, if $L_1 \succ L_2 \succ L_3$,
then there exists $a \in (0, 1)$
such that $[L_1 : a,\ L_3 : (1 - a)] \sim L_2$

Independence

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$$L_1 \succ L_2 \text{ if, and only if, } [L_1 : a,\ L_3 : (1-a)] \succ [L_2 : a,\ L_3 : (1-a)].$$

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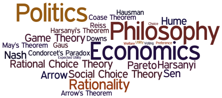
$$L_1 \sim L_2 \text{ if, and only if, } [L_1 : a,\ L_3 : (1-a)] \sim [L_2 : a,\ L_3 : (1-a)].$$

Axioms



Preference $\succeq$ is transitive and complete

Compound Lotteries The decision maker is indifferent between every compound lottery and the *corresponding* simple lottery.

Independence For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0,1]$, $L_1 \succ L_2$

if, and only if,

$[L_1 : a,\ L_3 : (1-a)] \succ [L_2 : a,\ L_3 : (1-a)]$.

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$u : \mathcal{L} \to \Re$ is linear provided for all $L = [L_1 : p_1, \ldots, L_n : p_n] \in \mathcal{L}$,

$$u(L) = \sum_{i=1}^{n} p_i u(L_i)$$

von Neumann-Morgenstern Representation Theorem A binary relation $\succeq$ on $\mathcal{L}$ satisfies Preference, Compound Lotteries, Independence and Continuity if, and only if, $\succeq$ is representable by a linear utility function $u : \mathcal{L} \to \Re$.

Moreover, $u' : \mathcal{L} \to \Re$ represents $\succeq$ iff there exists real numbers $c > 0$ and d such that $u'(\cdot) = cu(\cdot) + d$. ("$u$ is unique up to linear transformations.")