## PHIL309P

# Methods in Philosophy, Politics and Economics 

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## Ordinal Utility Theory

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A utility function on a set $X$ is a function $u: X \rightarrow \mathbb{R}$

## Ordinal Utility Theory

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A utility function on a set $X$ is a function $u: X \rightarrow \mathbb{R}$

Fact. Suppose that $X$ is finite and $\succeq$ is a complete and transitive ordering over $X$, then there is a utility function $u: X \rightarrow \mathfrak{R}$ that represents $\succeq$ (i.e., $x \succeq y$ iff $u(x) \geq u(y)$ )

## Ordinal Utility Theory

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Fact. Suppose that $X$ is finite and $\succeq$ is a complete and transitive ordering over $X$, then there is a utility function $u: X \rightarrow \mathfrak{R}$ that represents $\succeq$
(i.e., $x \succeq y$ iff $u(x) \geq u(y)$ )

Utility is defined in terms of preference (so it is an error to say that the agent prefers $x$ to $y$ because she assigns a higher utility to $x$ than to $y$ ).

## Important

All three of the utility functions represent the preference $x \succ y \succ z$

| Item | $u_{1}$ | $u_{2}$ | $u_{3}$ |
| :---: | :---: | :---: | :---: |
| $x$ | 3 | 10 | 1000 |
| $y$ | 2 | 5 | 99 |
| $z$ | 1 | 0 | 1 |

$x \succ y \succ z$ is represented by both $(3,2,1)$ and $(1000,999,1)$, so one cannot say that $y$ is "closer" to $x$ than to $z$.

## Lotteries

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Suppose that $X$ is a set of outcomes.

A (simple) lottery over X is denoted $\left[x_{1}: p_{1}, x_{2}: p_{2}, \ldots, x_{n}: p_{n}\right]$ where for $i=1, \ldots, n, x_{i} \in X$ and $p_{i} \in[0,1]$, and $\sum_{i} p_{i}=1$.

Let $\mathcal{L}$ be the set of (simple) lotteries over $X$. We identify elements $x \in X$ with the lottery $[x: 1]$.

## Lotteries

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Suppose that $X=\left\{x_{1}, \ldots, x_{n}\right\}$ is a set of outcomes. A lottery over $X$ is a tuple $\left[x_{1}: p_{1}, x_{2}: p_{2}, \ldots, x_{n}: p_{n}\right]$ where $\sum_{i} p_{i}=1$.

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Let $\mathcal{L}$ be the set of lotteries.

## Expected monetary value

Suppose that the outcomes of a lottery are monetary values. So, $L=\left[x_{1}: p_{1}, x_{2}: p_{2}, \ldots, x_{n}: p_{n}\right]$, where each $x_{i}$ is an amount of money. Then,

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E V(L)=\sum_{i} p_{i} \times x_{i}
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E.g., if $L=[\$ 100: 0.55, \$ 50: 0.25, \$ 0: 0.20]$, then

$$
E V(L)=0.55 * 100+0.25 * 50+0.2 * 0=80
$$

## Problems with using monetary payoffs

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- Overly Restrictive: We care about more things than money.


## Problems with using monetary payoffs

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- The St. Petersburg Paradox: Consider the following wager: I will flip a fair coin until it comes up heads; if the first time it comes up heads is the $n^{\text {th }}$ toss, then I will pay you $2^{n}$. What's the most you'd be willing to pay for this wager? What is its expected monetary value?


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- Valuing Money: Doesn't the value of a wager depend on more than merely how much it's expected to pay out? (I.e., your total fortune, how much you personally care about money, etc.)


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- Valuing Money: Doesn't the value of a wager depend on more than merely how much it's expected to pay out? (I.e., your total fortune, how much you personally care about money, etc.)
- Risk-aversion: Is it irrational to prefer a sure-thing $\$ x$ to a wager whose expected payout is $\$ x$ ?

We should move away from "monetary payouts" to "utility".

## Expected Utility

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Suppose that $X=\left\{x_{1}, \ldots, x_{n}\right\}$ and $u: X \rightarrow \mathbb{R}$ is a utility function on $X$.

This can be extended to an expected utility function $E U: \mathcal{L}(X) \rightarrow \mathbb{R}$ where

$$
\begin{aligned}
E U\left(\left[x_{1}: p_{1}, \ldots, x_{n}: p_{n}\right]\right) & =p_{1} \times u\left(x_{1}\right)+\cdots+p_{n} \times u\left(x_{n}\right) \\
& =\sum_{i=1}^{n} p_{i} \times u\left(x_{i}\right)
\end{aligned}
$$

Suppose that Ann is faced with the choice between lotteries $L_{1}$ and $L_{2}$ where:

$$
L_{1}=\left[o_{1}: 0, o_{3}: 0.25, o_{3}: 0.75\right] \quad L_{2}=\left[o_{1}: 0.2, o_{2}: 0, o_{3}: 0.8\right]
$$

Can expected utility theory tell us how Ann should rank $L_{1}$ and $L_{2}$ ?

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Can expected utility theory tell us how Ann should rank $L_{1}$ and $L_{2}$ ? No!
Suppose that Ann is also faced with the choice between lotteries $L_{3}$ and $L_{4}$ where:

$$
L_{3}=\left[o_{1}: 0.8, o_{2}: 0, o_{3}: 0.2\right] \quad L_{4}=\left[o_{1}: 0, o_{2}: 1, o_{3}: 0\right]
$$

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If we know that Ann ranks $L_{1}$ over $L_{2}$ (e.g., $L_{1} \succ L_{2}$ ), can we conclude anything about how Ann ranks $L_{3}$ and $L_{4}$ ?

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If we know that Ann ranks $L_{1}$ over $L_{2}$ (e.g., $L_{1} \succ L_{2}$ ), can we conclude anything about how Ann ranks $L_{3}$ and $L_{4}$ ? Yes: Ann must rank $L_{4}$ over $L_{3}$ (e.g., $L_{4} \succ L_{3}$ ).

## Cardinal Utility Theory

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$$
u: X \rightarrow \mathbb{R}
$$

Which comparisons are meaningful?

1. $u(x)$ and $u(y)$ ? (ordinal utility)

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2. $u(x)-u(y)$ and $u(a)-u(b)$ ?

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u: X \rightarrow \mathbb{R}
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Which comparisons are meaningful?

1. $u(x)$ and $u(y)$ ? (ordinal utility)
2. $u(x)-u(y)$ and $u(a)-u(b)$ ?
3. $u(x)$ and $2 * u(z)$ ?

## Ordinal vs. Cardinal Utility

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## Ordinal vs. Cardinal Utility

Ordinal scale: Qualitative comparisons of objects allowed, no information about differences or ratios.

## Cardinal scales:

Interval scale: Quantitative comparisons of objects, accurately reflects differences between objects.
E.g., the difference between $75^{\circ} \mathrm{F}$ and $70^{\circ} \mathrm{F}$ is the same as the difference between $30^{\circ} \mathrm{F}$ and $25^{\circ} \mathrm{F}$ However, $70^{\circ} \mathrm{F}\left(=21.11^{\circ} \mathrm{C}\right)$ is not twice as hot as $35^{\circ} \mathrm{F}\left(=1.67^{\circ} \mathrm{C}\right)$.

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Ratio scale: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10 lb ( $=4.53592 \mathrm{~kg}$ ) is twice as much as 5 lb ( $=2.26796 \mathrm{~kg}$ ).

## Cardinal Utility Theory

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$x \succ y \succ z$ is represented by both $(3,2,1)$ and $(1000,999,1)$, so we cannot say $y$ whether is "closer" to $x$ than to $z$.

Key idea: Ordinal preferences over lotteries allows us to infer a cardinal (interval) scale (with some additional axioms).

John von Neumann and Oskar Morgenstern. The Theory of Games and Economic Behavior. Princeton University Press, 1944.

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## A Choice



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$$
[1: B] \sim[p: R, 1-p: S]
$$

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$1 * u(B)=p * u(R)+(1-p) * u(S)$

## A Choice



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R
B
W
$S$


$$
u(B)=p * 1+(1-p) * 0=p
$$

## Lotteries

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Let $\mathcal{L}$ be the set of lotteries. Suppose that $\succeq \subseteq \mathcal{L} \times \mathcal{L}$ is a preference ordering on $\mathcal{L}$.

## Reduction of Compound Lotteries

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Reduction of Compound Lotteries: If the prize of a lottery is another lottery, then this can be reduced to a simple lottery over prizes.

This eliminates utility from the thrill of gambling and so the only ultimate concern is the prizes.


## Continuity

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Continuity For all $L_{1}, L_{2}, L_{3} \in \mathcal{L}$, if $L_{1} \succ L_{2} \succ L_{3}$, then there exists $a \in(0,1)$ such that $\left[L_{1}: a, L_{3}:(1-a)\right] \sim L_{2}$

## Independence

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Independence $\quad$ For all $L_{1}, L_{2}, L_{3} \in \mathcal{L}$ and $a \in(0,1]$,

$$
L_{1} \succ L_{2} \text { if, and only if, }\left[L_{1}: a, L_{3}:(1-a)\right] \succ\left[L_{2}: a, L_{3}:(1-a)\right] .
$$

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$$

$$
L_{1} \sim L_{2} \text { if, and only if, }\left[L_{1}: a, L_{3}:(1-a)\right] \sim\left[L_{2}: a, L_{3}:(1-a)\right] .
$$

## Axioms

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Preference

Independence

Continuity

Compound Lotteries The decision maker is indifferent between every compound lottery and the corresponding simple lottery.
$\succeq$ is transitive and complete

For all $L_{1}, L_{2}, L_{3} \in \mathcal{L}$ and $a \in(0,1], L_{1} \succ L_{2}$ if, and only if,
$\left[L_{1}: a, L_{3}:(1-a)\right] \succ\left[L_{2}: a, L_{3}:(1-a)\right]$.
For all $L_{1}, L_{2}, L_{3} \in \mathcal{L}$ and $a \in(0,1]$, if $L_{1} \succ L_{2} \succ L_{3}$, then there exists $a \in(0,1)$ such that $\left[L_{1}: a, L_{3}:(1-a)\right] \sim L_{2}$
$u: \mathcal{L} \rightarrow \Re$ is linear provided for all $L=\left[L_{1}: p_{1}, \ldots, L_{n}: p_{n}\right] \in \mathcal{L}$,

$$
u(L)=\sum_{i=1}^{n} p_{i} u\left(L_{i}\right)
$$

von Neumann-Morgenstern Representation Theorem A binary relation $\succeq$ on $\mathcal{L}$ satisfies Preference, Compound Lotteries, Independence and Continuity if, and only if, $\succeq$ is representable by a linear utility function $u: \mathcal{L} \rightarrow \Re$. Moreover, $u^{\prime}: \mathcal{L} \rightarrow \Re$ represents $\succeq$ iff there exists real numbers $c>0$ and $d$ such that $u^{\prime}(\cdot)=c u(\cdot)+d$. (" $u$ is unique up to linear transformations.")

