Sets, Relations and Functions

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An important part of this course One aspect of this course that many students struggle with is mathematical notation.

1 Basic Set Theory

We often groups things together. Everyone in this class, your group of friends, your family. These are all collections of people. Set theory is a mathematical language to reason about collections.

Set

Any collection of objects. It is assumed that there is a *universal set*, called the *domain of discourse*, so that a set is a collection of objects from the universal set.

In these notes, sets are denoted with upper-case letters, and elements of sets with lower-case letters.

Remark 1.1 Note that there is no standard notation for sets and elements. For instance, some texts may use lowercase letters to denote sets.

There are two ways to write down a set:

1. List all the elements of the set. Each element should be separated by a comma and there entire list of elements is written between curly brackets: '}' and '{'. For example, the set that contains the first 5 letters of the alphabet is $\{a, b, c, d, e\}$.

2. Define a property that *all* objects in the set have in common. For example, the set of all positive integers is $A = \{x \mid x \geq 0 \text{ and } x \text{ is an integer}\}$. This is read "the set of all x such that x is an integer that is greater than or equal to zero".

Element

A member of a set. When x is an element of the set A, we write $x \in A$.

Subset

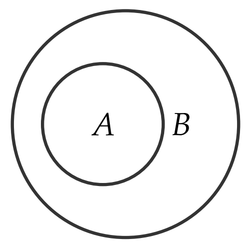
A set A is a **subset** of a set B, denoted $A \subseteq B$, provided that every element of A is also an element of B.

We use the phrase "...is contained in" when talking about both elements and subsets. If $A = \{1, 2, 3\}$, then it is common to say that "A contains 1" or "1 is contained in A". Something that can be confusing for beginners is that it is also common to say that "the set $\{1, 2\}$ is contained in A". It is important to remember that the subset notation "$A \subseteq B$" should only be used when A and B are both sets. For example, if $X = \{a, b, c\}$, then it is incorrect to write "$a \subseteq X$" since a is not a set. Note that it is possible that a set contains an element that is also a subset. This can happen when sets contain other sets as members. For instance, the set $X = \{a, b, c\}$ contains three elements (i.e., $a \in X$, $b \in X$ and $c \in X$), and $Y = \{a, \{b, c\}\}$ contains two elements ($a \in Y$ and $\{b, c\} \in Y$). An example of a set that has an element that is also a subset is $Z = \{a, \{a\}\}$, since $\{a\} \subseteq Z$ (since $a \in Z$) and $\{a\} \in Z$.

Visualizing sets

A *Venn diagram* is a geometrical visualization of a set, or collection of sets. For instance, $A \subseteq B$ can be depicted as follows:

$A \subseteq B$



Operations on sets

We will discuss a number of operations on sets. That is, ways of combining or modifying sets to form new sets.

Union

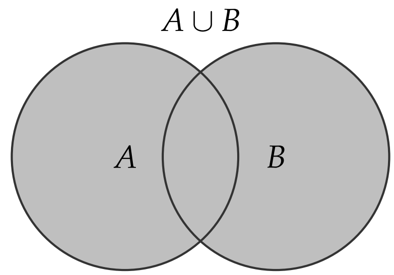
The **union** of two (or more) sets is a set that contains all the elements of each set. For two sets A and B, the union of A and B, denoted $A \cup B$, is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

More generally, if $\mathcal{S}$ is any collection of sets, then

$$\bigcup \mathcal{S} = \{x \mid x \in A \text{ for some } A \in \mathcal{S}\}.$$

The union of two sets can be pictured as follows (the gray shaded region is the union of A and B):



Example: Union

- If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then $A \cup B = \{1, 2, 3, a, b\}$
- If $A = \{1, 2, 3\}$ and $B = \{1, 3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$
- If $A = \{1, 2, 3\}$ and $B = \{\{1, 3\}, 4\}$, then $A \cup B = \{1, 2, 3, \{1, 3\}, 4\}$

Exercise 1.2 *Use a Venn diagram to convince yourself of the following two facts:*

- *For any sets A and B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.*
- *For any sets A and B, $A \subseteq B$ if, and only if, $A \cup B = B$.*

Intersection

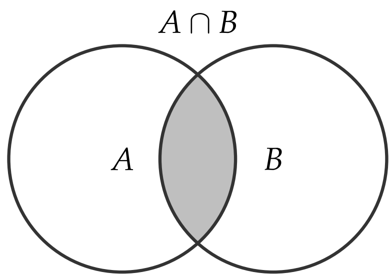
The **intersection** of two (or more) sets is the set of all items in common to each set. If A and B are two sets, then the intersection of A and B, denoted $A \cap B$, is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

More generally, if $\mathcal{S}$ is any collection of sets, then

$$\bigcap \mathcal{S} = \{x \mid x \in A \text{ for all } A \in \mathcal{S}\}.$$

The intersection can be pictured as follows (the gray shaded region is the intersection of A and B):



Example: Union

- If $A = \{1, 2, 3\}$ and $B = \{1\}$, then $A \cap B = \{1\}$.
- If $A = \{1, 2, 3\}$ and $B = \{\{1, 3\}, 2\}$, then $A \cap B = \{2\}$.
- If $A = \{1, 2\}$ and $B = \{3, 4\}$, then $A \cap B$ contains no elements (it is the empty set).

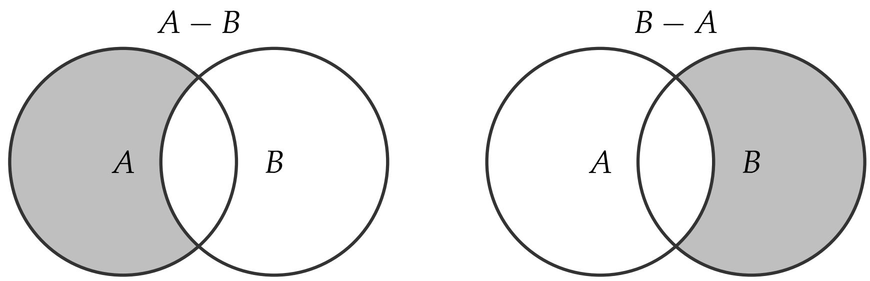
Exercise 1.3 *Use a Venn diagram to convince yourself of the following two facts:*

- *For any sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.*
- *For any sets A and B, $A \subseteq B$ if, and only if, $A \cap B = A$.*

Set Difference

The difference between two sets A and B (A minus B), denoted $A - B$ is all the elements in A but not in B. The difference between A and B is the set $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

The differences between A and B can be pictured as follows:



Complement

The **complement** of a set is the set of all elements not contained in that set. Formally, the complement of the set A is

$$A^C = \{x \mid x \text{ is in the universal set and } x \notin A\}.$$

Example: Set Difference and Complement

- If $A = \{1, 2, 3\}$ and $B = \{1, 2, 4\}$, then $A - B = \{3\}$ and $B - A = \{4\}$.
- If $A = \{1, 2, 3\}$ and $B = \{1\}$, then $A - B = \{2, 3\}$ and $B - A$ is the empty set.
- If the universal set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 2, 3\}$, then $A^C = \{0, 4, 5, 6, 7, 8, 9\}$.

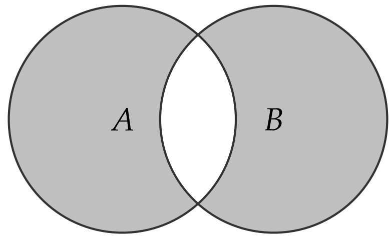
Exercise 1.4 *Using Venn diagrams, convince yourself that for any sets A and B,* $A - B = A \cap B^C$.

Symmetric Difference

The symmetric difference of two sets is all the elements in either set *but not in both*. The symmetric difference is the set $(A - B) \cup (B - A)$.

The symmetric difference is pictured as follows:

$(A - B) \cup (B - A)$



Example: Symmetric Difference

- The symmetric difference of $A = \{1,2,3\}$ and $B = \{1,2,4\}$ is $\{3,4\}$.
- The symmetric difference $A = \{1,2,3\}$ and $B = \{1\}$ is $\{2,3\}$.

We conclude this brief introduction to set theory by defining some important sets.

Empty Set

The empty set, or null set, is a set that contains no elements. We write $\emptyset$ to denote the set containing no elements.

Power Set

The **power set** is the set of all subsets. If A is a set, then the power set of A is the set $\wp(A) = \{B \mid B \subseteq A\}$.

Example: Finding the power set

Suppose that $A = \{1, 2, 3\}$. Then the power set of A is defined as follows:

$$\wp(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Cardinality of a Set

The cardinality of a finite[a] set A is the total number of elements in A, and is denoted $|A|$.

[a] The notion of cardinality can be applied to infinite sets as well. However, a discussion of this is beyond the scope of these introductory notes.

Exercise 1.5

1. *What is the powerset of $\emptyset$ (i.e., $\wp(\emptyset)$)?*
2. *If A has n elements, $|A| = n$, then how many elements are in $\wp(A)$?*

2 Relations

The order in which we list elements in a set does not matter. That is, $\{a, b\}$ is the same set as $\{b, a\}$ (they both denote the set consisting of two elements 'a' and 'b'). There are many situations in which the order in which the elements appear is important. When the order in which the elements appear matters, we use '(' and ')'. For example, (a, b) is an **ordered pair**, or **tuple**, of length 2. The first component is a and the second component is b. Since the order in which the elements appear matters, we have that $(a, b) \neq (b, a)$. More generally, examples of **tuples** of length 5 consisting of elements from the set $\{a, b, c, d, e\}$ include (a, b, c, d, e), (b, d, a, d, e) or (a, a, b, b, e) (note that in the last tuple we allow the same element to be repeated.

Product

Suppose that A and B are non-empty sets. The **product** of A and B, denoted $A \times B$, is the set of ordered pairs where the first component comes from A and the second component comes from B. That is,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example: Products on sets

The set $X \times X$ is the **cross-product** of X with itself. That is, it is the set of all pairs of elements (called **ordered pairs**) from X. For example, if $X = \{a, b, c\}$, then

$$X \times X = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}.$$

Suppose that X is a set and that n a positive integer. We write X^n for the **n-fold product** of X. That is $X^1 = X$ and for all $n > 1$, $X^{n+1} = X \times X^n$. So, $X^2 = X \times X$ and X^3 is the set of all tuples from X of length 3 (formally, $X^3 = X \times (X \times X)$).

A **relation** R on a set X is a subset of $X \times X$ (the set of pairs of elements from X). Formally, R is a relation on X means that $R \subseteq X \times X$. It is often convenient to write $a\ R\ b$ for $(a, b) \in R$. To help appreciate this definition, consider the following examples. Suppose that X is the set of people in a room and that everyone in the room is pointing at some person in the room. A relation can be used to describe who is pointing at whom, where for $a, b \in X$, $a\ R\ b$ means that person a is pointing at person b. A second example is the“taller-than" relation, denoted $T \subseteq X \times X$, where $a\ T\ b$ means that person a is taller than person b. Typically, we are interested in relations satisfying special properties.

Properties of relations

Suppose that $R \subseteq X \times X$ is a relation.

- R is **reflexive** provided that for all $a \in X$, $a\ R\ a$.

- R is **irreflexive** provided that for all $a \in X$, it is not the case that $a\ R\ a$.
- R is **complete** provided that for all $a, b \in X$, $a\ R\ b$ or $b\ R\ a$ (or both).
- R is **symmetric** provided that for all $a, b \in X$, if $a\ R\ b$ then $b\ R\ a$.
- R is **asymmetric** provided that for all $a, b \in X$, if aRb then not-$b\ R\ a$.
- R is **anti-symmetric** provided that for all $a, b \in X$ if $a\ R\ b$ and $b\ R\ a$, then $a = b$.
- R is **transitive** provided that for all $a, b, c \in X$, if $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$.

Remark 2.1 As stated, completeness implies reflexivity (let $a = b$ in the above definition of completeness). Often, completeness is defined as follows: for all *distinct* $a, b \in X$, $a\ R\ b$ or $b\ R\ a$. In what follows, we will use the above stronger definition of completeness where completeness implies reflexivity.

Recall the example of a relation R that describes people pointing at other people in the room. If R is reflexive, then this means everyone is pointing at themselves. If R is irreflexive, then this means that no-one is pointing at themselves. This example illustrates the fact that irrelexivity is *not* the negation of reflexivity. That is, there are examples of relations that are neither reflexive nor irreflexive. If R is complete, then this means that every person in the room is either pointing at somebody or being pointed at. Symmetry of R means that every person that is being pointed at is pointing back at the person pointing at them. Asymmetry of R means that nobody is pointing back at the person pointing at them. Similar to the relationship between reflexivity and irreflexivity, asymmetry is *not* the negation of symmetry. Picturing transitivity of the relation R is a bit more complicated. If the relation R is transitive, then everyone is pointing at the person that is being pointed to by the person that they are pointing at.

Exercise 2.2 *Suppose that there are 5 people in a room. Draw a picture of a situation where the people are pointing at each other and the relation that describes the situation* *is transitive.*

Exercise 2.3 *What properties does the "better-than" relation satisfy?*

Remark 2.4 (Describing Relations) Suppose that $R \subseteq X \times X$ is a relation. We will often use the following shorthand to denote elements in the relation: If $x_1, \ldots, x_n \in X$, then

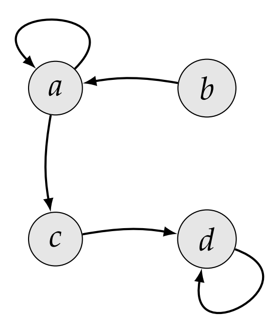
$$x_1 \; R \; x_2 \; R \cdots x_{n-1} \; R \; x_n$$

means that for all $i = 1, \ldots, n-1$, $(x_i, x_{i+1}) \in R$ or $(x_i, x_j) \in R$ for all $j < i$ if R is assumed to be transitive (or $j \leq i$ if R is assumed to also be reflexive). For example, if R is transitive and reflexive, then $a \; R \; b \; R \; c$ means that $\{(a,a), (a,b), (b,b), (a,c), (b,c), (c,c)\} \subseteq R$.

When thinking about relations, it is often helpful to draw a picture of the relation. For instance, suppose that $X = \{a, b, c, d\}$ and $R \subseteq X \times X$ is:

$$R = \{(a,a), (b,a), (c,d), (a,c), (d,d)\}.$$

Relations on X can be visualized as follows: Write down all the elements of X and draw an arrow from element x to element y when $(x,y) \in R$. Following this convention, the following pic:

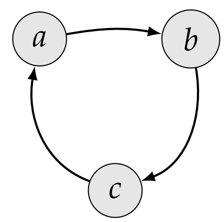


Cycle

A **cycle** in a relation $R \subseteq X \times X$ is a set of distinct elements $x_1, x_2, \ldots, x_n \in$

X such that for all $i = 1, \dots, n-1$, $x_i \; R \; x_{i+1}$, and $x_n \; R \; x_1$. A relation R is said to be **acyclic** if there are no cycles.

For example, suppose that $X = \{a, b, c\}$ and $R = \{(a, b), (b, c), (c, a)\}$. Then R is a cycle on X which can be depicted as:



Exercise 2.5 *Suppose that X has three elements (i.e., $X = \{a, b, c\}$. How many cycles can be formed from elements in X?*

Often one is interested in relations that are acyclic. The main reason is that if a relation on X has a cycle, then there is a subset of $S \subseteq X$ for which there is no element of S that can be considered the "largest" or "best" in S according to R. There are two notions of the "largest" or "best" element in a set according to a relation R.

Maximal

Suppose that X is a set, $S \subseteq X$ and R is a relation on X. An element $m \in S$ is R**-maximal** in S provided that for all $x \in S$, if $x \; R \; m$, then $x = m$. Let $\max_R(S)$ be the set of maximal elements of S.

An equivalent definition of an element $m \in S$ being R**-maximal** in S is that there is no $x \in S$ such that $x \; R \; m$ (except possibly m). So, an element $m \in S$ is maximal means that there is no element of S that is R-related to m.

Example: Examples of maximal elements

Suppose that $X = \{a, b, c\}$. Consider the following examples

- If $R = \{(a,b),(b,c),(a,c)\}$, then $\max_R(X) = \{a\}$ (there is no $x \in X$ such that $x\ R\ a$).
- If $R = \{(a,b),(b,c),(c,a)\}$, then $\max_R(X) = \emptyset$ (for each $x \in X$, there is $y \in X$ such that $x \neq y$ and $y\ R\ x$).
- $R = \emptyset$, then $\max_R(X) = \{a, b, c\}$ (for each $x \in X$, there is no $y \in X$ such that $y\ R\ x$).
- $R = \{(a,a),(b,b),(c,c)\}$, then $\max_R(X) = \{a, b, c\}$ (for each $x \in X$, there is no $y \in X$ such that $x \neq y$ and $y\ R\ x$).
- $R = \{(a,b),(c,b)\}$, then $\max_R(X) = \{a, c\}$ (for each $x \in \{a, c\}$, there is no $y \in X$ such that $x \neq y$ and $y\ R\ x$).
- If $R = \{(a,b)\}$, then $\max_R(X) = \{a, c\}$ (note that there is no element that is R-related to c)
- If $R = \{(a,b),(b,a)\}$, then $\max_R(X) = \{c\}$ (again, note that there is no element that is R-related to c).

The above examples illustrate that given a relation R on a set S, there may be no maximal elements in S, a unique maximal element of S, or more than one maximal element in S. However, if a relation R is complete on S (i.e., all elements in S are related by R in some way), then there can be at most one maximal element. Furthermore, when there is a unique maximal element in a set S for a relation R, this element is R-related to every element of S. [1] This motivates the following definition.

[1] Unless S contains a single element and R is empty. In this degenerate case, the only element of S is maximal.

Maximum

Suppose that X is a set, $S \subset X$ and R is a relation on X. An element $m \in S$ is the R-**maximum** of S provided that for each $x \in S$, either $m\ R\ x$ or $x = m$.

So, an element is the maximum of S provide that it is R-related to every other element in S. Note that if a set S has an R-maximum, then it must be unique. Every maximum element is maximal, but Example 2 shows that there may be maximal elements that are not maximum.

Example: Examples of maximum elements

Suppose that $X = \{a, b, c\}$. Consider the following examples:

- If $R = \{(a, b), (b, c), (a, c)\}$, then a is the maximum in X.
- If $R = \{(a, b), (b, c), (c, a)\}$, then there is no maximum in X.
- $R = \{(a, b), (c, b)\}$, then there is no maximum in X.
- If $R = \{(a, b)\}$, then there is no maximum in X (but a is the maximum of $Y = \{a, b\}$).
- If $R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$, then a is the maximum in X.

Exercise 2.6 *Is it possible to find a relation that has a cycle and a non-empty set of maximal elements? What about a relation that has a cycle, a non-empty set of maximal elements, and is complete and transitive?*

Exercise 2.7 *Prove that if R is acyclic, then* $\max_R(Y) \neq \emptyset$. *Is the converse true? (Why or why not?)*

Relations are an important mathematical tool used throughout Economics, Logic and Philosophy. You have already studied binary relations during your mathematical eduction: $=, \leq, \geq, <$, and $>$ are all relations on numbers (eg., the natural numbers $\mathbb{N}$, real numbers $\mathbb{R}$, rational numbers $\mathbb{Q}$, etc.) and $\subseteq$ is a relation on the power set of a set S. For example, the binary relation $\leq \subseteq \mathbb{N} \times \mathbb{N}$ is the set

$$\{(a, b) \mid a, b \in \mathbb{N} \text{ and } a \text{ is less than or equal to } b\}$$

Exercise 2.8 *What properties do $\leq$, $<$, and $=$ satisfy (assume the relations are defined on the natural numbers $\mathbb{N}$)?*

If X is a set, we write X^n to denote the ***n*-fold cross product**. That is

$$X^n = \underbrace{X \times \cdots \times X}_{n\text{-times}}$$

For example, if $X = \{a, b\}$, then $X^2 = \{(a, a), (a, b), (b, a), (b, b)\}$.

Example: Products of products

There are situations when one must consider products of sets which are themselves products. For example, suppose that $X = \{a, b\}$. Let $Y = X^2$. Then, elements Y^2 consists of tuples of length 2 where each component is a tuple of length 2 of elements from X:

$$\begin{aligned} Y^2 \quad = \quad & \{((a,a),(a,a)),((a,a),(a,b)),((a,a),(b,a)),((a,a),(b,b)), \\ & ((a,b),(a,a)),((a,b),(a,b)),((a,b),(b,a)),((a,b),(b,b)), \\ & ((b,a),(a,a)),((b,a),(a,b)),((b,a),(b,a)),((b,a),(b,b)), \\ & ((b,b),(a,a)),((b,b),(a,b)),((b,b),(b,a)),((b,b),(b,b))\} \end{aligned}$$

An example of an element in Y^3 is $((a, a), (b, a), (a, b))$. Since Y has 4 elements, we have that $|Y^3| = 64$. More generally, if A has m elements, $|A| = m$, then $|A^n| = m^n$.

Often relations are intended to represent an *ordering* or *ranking* of a set of objects. This is the motivation behind the following definitions.

Preorder

A relation R on X is a **preorder** if, and only if, R is reflexive (for all $x \in X$, $x\ R\ x$) and transitive (for all $x, yz \in X$ if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$).

Partial Order

A relation R on X is a **partial order**, or **poset** if, and only if, R is reflexive (for all $x \in X$, $x\ R\ x$), anti-symmetric (for all $x, y \in X$, if $x\ R\ y$ and $y\ R\ x$ then $x = y$) and transitive (for all $x, yz \in X$ if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$).

Total Order

A partial order R on X is a **total order**, or **linear order**, if it is complete (for all $x, y \in X$, either $x\ R\ y$ or $y\ R\ x$). A total order R on X that satisfies asymmetry (for all $x, y \in X$, if $x\ R\ y$ then it is not the case that $y\ R\ x$) is called a **strict total order**, or **strict linear order**.

The standard example of a partial order is the relation $\subseteq$ on $\wp(X)$. An example of a total order is the less-than-or-equal relation on the natural numbers $\leq \subseteq \mathbb{N} \times \mathbb{N}$. Finally, an example of a strict total order is the less-than relation on the natural numbers $< \subseteq \mathbb{N} \times \mathbb{N}$.

Suppose that R is a relation on X. We say that R' is a **subrelation** of R when $R' \subseteq R$. For example, if $R \subseteq X \times X$, then we can define the following subrelations (typically, R is assumed to be preorder):

Strict Subrelation For each $x, y \in X$, P_R if, and only if, $x\ R\ y$ and it is not the case that $y\ R\ x$.

Indifference Order: For each $x, y \in X$, $x\ I_R\ y$ if, and only if, $x\ R\ y$ and $y\ R\ x$.

Incomparability Order: For each $x, y \in X$, $x\ N_R\ y$ if, and only if, it is not the case that $x\ R\ y$ and it is not the case that $y\ R\ x$

The above relations play an important role when R is intended to represent a ranking or preference ordering of a decision maker.

Exercise 2.9 *If R is complete and transitive, what properties do P_R, I_R and N_R satisfy?*

Equivalence Relation

A relation R that is reflexive, symmetric and transitive is said to be an **equivalence relation**

Equivalence Class

If R is an equivalence relation on A, then for each $a \in A$, the equivalence class of a, denoted by $[a]$, is the following set $[a] = \{b \mid aRb\}$.

Example: Equivalence Relation

Suppose that $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$. Then R is an equivalence relation and we have that $[1] = [2] = \{1, 2\}$ and $[3] = \{3\}$.

Suppose that P is the partial order

$$P = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (2, 3), (3, 3)\}.$$

Then the indifference relation associated with R,

$$I_P = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

is an equivalence relation on A.

We can now state our first theorem. It is somewhat technical, but illustrates a fundamental idea about equivalence classes. Every equivalence relation *par-*

titions a set and every partition of a set has an equivalence relation associated with it. We start by defining a **partition**.

Partition

A **partition** of a set S is a collection of subsets of S, $\mathcal{S} = \{S_1, S_2, \ldots\}$ (possibly infinite), such that

- the sets are **pairwise disjoint**: if $S_i, S_j \in \mathcal{S}$ with $i \neq j$, then $S_i \cap S_j = \emptyset$
- their union is S, that is, $S = \cup_{S_i \in \mathcal{S}} S_i$.

Theorem 2.10 *The equivalence classes of any equivalence relation R on a set A forms a partition of A, and any partition of A determines an equivalence relation on A for which the sets in the partition are the equivalence classes.*

Proof. Suppose R is an equivalence relation on A. We must show that the equivalence classes of R forms a partition of A.

1. Each equivalence class is non-empty, since $a\ R\ a$ for all $a \in A$.
2. Clearly A is the union of all the equivalence classes (since each element of A belongs to at least one equivalence class)
3. We must show any two equivalence classes are disjoint. Let $[a], [b]$ be two distinct equivalence classes. Suppose $c \in [a] \cap [b]$. Then $a\ R\ c$ and $b\ R\ c$. Hence by symmetry, $c\ R\ b$. And so by transitivity, $a\ R\ b$.

   Let $x \in [a]$, then $x\ R\ c$ and by the above argument $x\ R\ b$ (Why?), and so $x \in [b]$. Thus $[a] \subseteq [b]$. Using a similar argument, we can show $[b] \subseteq [a]$. Therefore $[a] = [b]$, which contradicts the fact that $[a]$ and $[b]$ are *distinct* equivalence classes.

For the second part of the theorem, suppose $\mathcal{A} = \{A_1, \ldots, A_n\}$ is any partition of A. Define $R = \{(a, b) \mid a \in A_i \text{ and } b \in A_i\}$. We must show that R is reflexive. Let $a \in A$ be any element. Then $a \in A_i$ for some i, and hence by definition of R, $a\ R\ a$. Next we will show that R is symmetric. Suppose $a\ R\ b$. Then $a \in A_i$ and $b \in A_i$ for some i. Then clearly, $b \in A_i$ and $a \in A_i$ and hence $b\ R\ a$. We must show R is transitive. Suppose, $a\ R\ b$ and $b\ R\ c$. Then $a \in A_i$ and $b \in A_i$, and $b \in A_j$ and $c \in A_j$ for some i, j. Since $b \in A_i \cap A_j$, $A_i = A_j$ (since the elements of $\mathcal{A}$ are pairwise disjoint). Therefore, $a \in A_i$ and $c \in A_i$ and hence $a\ R\ c$. QED

Maximal/Maximum elements reconsidered: When R is a complete complete relation, then it is convenient to define the maximal elements, denoted $\text{best}_R(X)$, as follows:

$$\text{best}_R(X) = \{x \in X \mid x\ R\ y \text{ for all } y \in X\}$$

If R is anti-symmetric, then this definition is equivalent to Definition 2. To illustrate the difference between the definitions, suppose that $X = \{a, b, c\}$ and $R = \{(a, b), (b, a), (a, c), (b, c), (a, a), (b, b), (c, c)\}$. According to Definition 2 there are no maximal elements (for each element of $x \in X$, there is an element $y \in X$ such that $x \neq y$ and $y\ R\ x$). (Similarly, there is no maximum element in X). However, if R represents a ranking or preference order of a decision maker, then it is natural to interpret the fact that $a\ R\ b$ and $b\ R\ a$ as meaning that "a and b are *tied* according to R". According to the above definition, we have that $a, b \in \text{best}_R(X)$.

We can related the different notions of "best" elements of X as follows. Suppose that $X = \{a, b, c\}$ and R is a preorder on X. Then, let $X_I = \{\{a, b\}, \{c\}\}$ and define $\overline{R} \subseteq \overline{X} \times \overline{X}$ is as follows: For $Y, Z \in \overline{X}$, $Y\ \overline{R}\ Z$ iff for all $y \in Y$, and for all $z \in Z$, $y\ R\ z$. Now, $\overline{R}$ is an anti-symmetric, complete and transitive. Furthermore, we have that $\text{best}_R(X)$ is the maximum of $\max_{\overline{R}}(\overline{X})$.

More generally, Given a set X with a preorder R on X, let $\overline{X}$ be the set of equivalence classes according to the indifference relation I_R. Then, for $[x], [y] \in \overline{X}$, let $[x]\ \overline{R}\ [y]$ iff $x\ R\ y$. Then, $\overline{R}$ is a total order on $\overline{X}$ and $\text{best}_R(X)$ is the $\overline{R}$-maximum of $\overline{X}$.

2.1 Functions

A function from a set A to a set B is a way of associating elements of A with elements of B. Formally, a function is a special type of relation:

Function

A function f is a binary relation on A and B (i.e., $f \subseteq A \times B$) such that for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$. We write $f : A \to B$ when f is a function, and if $(a, b) \in f$, then write $f(a) = b$.

Example: Example of a function

Suppose that $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Examples of relations that are functions include:

- $R_1 = \{(1, a), (2, a), (3, b)\}$
- $R_2 = \{(1, a), (2, a), (3, a)\}$
- $R_3 = \{(1, a), (3, b)\}$

An example of a relation that is not a function is $R_4 = \{(1, a), (1, b), (3, b)\}$.

Suppose $f : A \to B$ is a function. The set A is called the **domain** and B the **codomain**.

Image

The image of a set $A' \subseteq A$ is the set:

$$f(A') = \{b \mid b = f(a) \text{ for some } a \in A'\}$$

Range

The **range** of a function is the image of its domain.

Surjection

f is a surjection (or onto) if its range is equal to its codomain. I.e., f is surjective iff for each $b \in B$, there exists an $a \in A$ such that $f(a) = b$

Injection

f is an injection (or 1-1) if distinct elements of the domain produce distinct elements of the codomain. I.e., f is 1-1 iff $a \neq a'$ implies $f(a) \neq f(a')$, or equivalently $f(a) = f(a')$ implies $a = a'$.

Bijection

f is a bijection if it is injective and surjective. In this case, f is often called a one-to-one correspondence.

Inverse Image

Suppose that $f : A \to B$ and that $Y \subseteq B$. The inverse image of Y is the set $f^{-1}(Y) = \{x \mid x \in A \text{ and } f(x) \in Y\}$

Example: Function on the powerset

Suppose that $X = \{a, b, c\}$. A function from non-empty subsets of X to non-empty subsets of X is denoted $f : (\wp(X) - \emptyset) \to (\wp(X) - \emptyset)$. An example of such a function is:

$$
\begin{array}{rcl}
f(\{a\}) & = & \{b\} \\
f(\{b\}) & = & \{b\} \\
f(\{c\}) & = & \{c\} \\
f(\{a,b\}) & = & \{a\} \\
f(\{a,c\}) & = & \{b\} \\
f(\{b,c\}) & = & \{b\} \\
f(\{a,b,c\}) & = & \{b,c\}
\end{array}
$$

Consider the following constraint

$$\text{for all } Y \in \wp(X) - \emptyset, f(Y) \subseteq Y$$

The above function does not satisfy this constraint since, for instance, $f(\{a\}) = \{b\} \not\subseteq \{a\}$ (we also have that $f(\{a,c\}) = \{b\} \not\subseteq \{a,c\}$. An example of a function that satisfies the above constraint is:

$$
\begin{array}{rcl}
f(\{a\}) & = & \{a\} \\
f(\{b\}) & = & \{b\} \\
f(\{c\}) & = & \{c\} \\
f(\{a,b\}) & = & \{a\} \\
f(\{a,c\}) & = & \{c\} \\
f(\{b,c\}) & = & \{b\} \\
f(\{a,b,c\}) & = & \{b,c\}
\end{array}
$$

3 Proofs

3.1 Introduction

Learning how to write mathematical proofs takes time and lots of practice. A proof of a mathematical statement is simply an explanation of that statement *written in the language of mathematics*. It is very important that you become comfortable with the definitions. If you don't know or understand the formal definitions, then you will not be able to write down your explanations. It would be like trying to explain something to someone in Italian without actually knowing Italian.

3.2 Proving Equality and Subset

How do you prove that two sets are equal? The answer to this question depends on who you are trying to convince. In this class, we will always err on the side of caution and given fairly detailed formal proofs. In turns out that proving two sets are equal reduces to proving the sets are subsets of each other.

Fact 3.1 *$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$*

Why is this true? Well, if A and B are equal, then they both name the same collection of objects. I.e., B is another name for the collection of objects that A names and vice versa. So, if A and B are equal then of course $A \subseteq B$ since A is always a subset of itself and B is simply another name for A. Similarly, we can show $B \subseteq A$. Conversely, suppose $A \subseteq B$ and $B \subseteq A$. We want to know that A and B name the same collection of objects. Suppose they didn't, then there should be some object $x \in A$ that is not in B *or* some object $y \in B$ that is not in A. Well, we know such an object x cannot exist since $A \subseteq B$, and so, every element of A is an element of B. Similarly, the element y cannot exist. Hence, A and B must name the same collection of objects.

What about trying to prove that two sets are *not* equal? This turns out to be easier. In order to show that A does not equal B, you need only find an element in A that is not in B **OR** an element of B that is not in A.

Showing two sets are equal reduces to proving that the sets are subsets of each other. But, how to show that a set is a subset of another set? The general procedure to show $A \subseteq B$ is to show that each element of A is also and element of B. This is straightforward if A and B are both finite sets. For example, suppose $A = \{2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. How do we show that $A \subseteq B$. Since A is finite, we simply notice that $2 \in B$, $3 \in B$ and $4 \in B$.

What if A is the set of even numbers and B is the set of all integers? We would get awfully tired (and bored) if we waited around to show that each and every element of A is also an element of B. Imagine A and B are two boxes, and you would like to know whether all the elements in A's box are also in B's box. Suppose you reach in box A and select an element, say the number 10. After inspecting 10, you notice that 10 is in fact an integer and so must also be an element of box B. But you are not satisfied, since you cannot be sure that the next element you choose from A will also be an element of B. In fact, even if you have shown that the first million even integers are all members of box B, you cannot be sure that the next element you select from box A will in fact be an integer. Instead, you should consider the *property* that all elements of A have in common and show that any object satisfying that property must be an element of B. What property does x satisfy if it is contained in A's box? The answer is $x = 2 \cdot n$, where n is some integer. Then, you simply notice that if n is an integer, then $2 \cdot n$ is also an integer; and hence, any element of A must also be an element of B.

3.3 Examples

Theorem 3.2 $\overline{A} \cup \overline{B} = \overline{A \cap B}$

Proof. We must show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ and $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

We will show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Suppose $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. Suppose $x \in \overline{A}$ then $x \notin A$. Then $x \notin A \cap B$ (if x is not in A then x is certainly not in both A and B). Hence $x \in \overline{A \cap B}$. Suppose $x \in \overline{B}$. For similar reason, $x \in \overline{A \cap B}$. Hence in either case, $x \in \overline{A \cap B}$. Therefore, $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

We must show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A \cap B}$. Then $x \notin A \cap B$, and so $x \notin A$ or[2] $x \notin B$. Hence either $x \in \overline{A}$ or $x \in \overline{B}$. In either case, $x \in \overline{A} \cup \overline{B}$.

QED

Theorem 3.3 *$A \subseteq B$ iff $A \cap B = A$.*

Proof. We must show $A \subseteq B$ implies $A \cap B = A$ and $A \cap B = A$ implies $A \subseteq B$.

Assume that $A \subseteq B$. We must show $A \cap B = A$. I.e. we must show (1) $A \cap B \subseteq A$ and (2) $A \subseteq A \cap B$. The first statement is trivial, it is always the

[2]Notice that $x \notin A \cap B$ **does not imply that** $x \notin A$ *and* $x \notin B$. The "and" in italics is should be an "or". Make sure you clearly understand the logic here, since this is often misunderstood by students.

case that $A \cap B \subseteq A$. For the second statement, assume $x \in A$. We must show $x \in A \cap B$. Since $A \subseteq B$, $x \in B$. Hence $x \in A \cap B$.

Assume $A \cap B = A$. We must show $A \subseteq B$. Let $x \in A$. Then $x \in A \cap B$ since $A = A \cap B$. Hence $x \in B$. QED

Attempt to answer these questions before looking at the answers.

Exercise 3.4 *Suppose that $f : A \to B$. Prove or disprove the following:*

1. *If $X \subseteq A$ and $Y \subseteq A$, then $f(X \cap Y) = f(X) \cap f(Y)$.*
2. *If $X \subseteq A$, $Y \subseteq A$ and f is 1-1,then $f(X \cap Y) = f(X) \cap f(Y)$.*
3. *If $X \subseteq B$ and $Y \subseteq B$, then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.*

Claim 3.5 *It not the case that if $X \subseteq A$ and $Y \subseteq A$, then $f(X \cap Y) \neq f(X) \cap f(Y)$.*

Proof of Claim 3.5. To prove this, we must find counterexample. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. And $f : A \to B$ be defined as follows: $f(1) = c$, $f(2) = b$ and $f(3) = c$. Let $X = \{1, 2\}$ and $Y = \{2, 3\}$. Then $X \cap Y = \{2\}$ and $f(X \cap Y) = f(\{2\}) = \{f(2)\} = \{b\}$. But, $f(X) \cap f(Y) = \{f(1), f(2)\} \cap \{f(2), f(3)\} = \{c, b\} \cap \{b, c\} = \{b, c\}$. Hence, $f(X \cap Y) \neq f(X) \cap f(Y)$.
QED (of Claim)

It is true that for any function $f : A \to B$ and all subsets $X, Y \subseteq A$, $f(X \cap Y) \subseteq f(X) \cap f(Y)$ (for a proof see below).

Claim 3.6 *If $X \subseteq A$, $Y \subseteq A$ and f is 1-1,then $f(X \cap Y) = f(X) \cap f(Y)$.*

Proof of Claim 3.6. Suppose that $f : A \to B$ is a 1-1 function. Let $X \subseteq A$ and $Y \subseteq A$. We must show (1) $f(X \cap Y) \subseteq f(X) \cap f(Y)$ and (2) $f(X) \cap f(Y) \subseteq f(X \cap Y)$.

Notice that (1) is true even if f is not 1-1. Let $y \in f(X \cap Y)$. Then there is an element $x \in X \cap Y$ such that $f(x) = y$. Since $x \in X \cap Y$, $x \in X$ and $x \in Y$. Therefore, $y = f(x) \in f(X)$ and $y = f(x) \in f(Y)$. Hence, $y \in f(X) \cap f(Y)$.

We now prove (2). Let $y \in f(X) \cap f(Y)$. Then $y \in f(X)$ and $y \in f(Y)$. Since $y \in f(X)$ there is $x_1 \in X$ such that $f(x_1) = y$. Since $y \in f(y)$, there is $x_2 \in Y$ such that $f(x_2) = y$. Since f is 1-1, $x_1 = x_2$. Therefore $x_1 = x_2 \in X \cap Y$; and so, $y = f(x_1) = f(x_2) \in f(X \cap Y)$. QED (of Claim)

Claim 3.7 *If $X \subseteq B$ and $Y \subseteq B$, then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.*

Proof of Claim 3.7. Let $f : A \to B$ be any function and suppose $X \subseteq B$ and $Y \subseteq B$. We must show $f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$ and $f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X \cap Y)$.

Suppose $x \in f^{-1}(X \cap Y)$. Then $f(x) \in X \cap Y$. Hence $f(x) \in X$ and $f(x) \in Y$. Since $f(x) \in X$, $x \in f^{-1}(X)$. Since $f(x) \in Y$, $x \in f^{-1}(Y)$. Therefore $x \in f^{-1}(X) \cap f^{-1}(Y)$.

Suppose $x \in f^{-1}(X) \cap f^{-1}(Y)$. Then $x \in f^{-1}(X)$ and $x \in f^{-1}(Y)$. Therefore, $f(x) \in X$ and $f(x) \in Y$. Hence, $f(x) \in X \cap Y$; and so, $x \in f^{-1}(X \cap Y)$. QED (of Claim)