Introduction to Many-Valued Modal Logic and Possibility Semantics

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Basic modal language: $\varphi \quad := \quad p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi$
where $p \in \mathsf{At}$

Frame: $\mathcal{M} = \langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$

Model: $\mathcal{M} = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a frame and
$V : \mathsf{At} \rightarrow \wp(W)$

Truth:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv, then $\mathcal{M}, v \models \varphi$

- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \rightarrow \psi$ iff if $\mathcal{M}, w \models \varphi$ then $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$

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$[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \rightarrow \wp(W)$, where for all $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathcal{M}} = \{v \mid \mathcal{M}, v \models \varphi\}$

Alternative Semantics

- General frames/models: $\langle W, R, \mathcal{A}\rangle$ where $\langle W, R\rangle$ is a frame, and $\mathcal{A} \subseteq \wp(W)$ is a BAO: Boolean algebra closed under the operator $m : \wp(W) \to \wp(W)$: where for all X, $m(X) = \{v \mid R(v) \subseteq X\}$.

  A general model is a structure $\langle W, R, \mathcal{A}, V\rangle$, where $\langle W, R, \mathcal{A}\rangle$ is a general frame and for all $p \in \text{At}$, $V(p) \in \mathcal{A}$.

- Neighborhood semantics: $\langle W, N, V\rangle$ where $N : W \to \wp(\wp(W))$

  $\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} = \{v \mid \mathcal{M}, v \models \varphi\} \in N(w)$

Language Extensions

- Multiple modalities: $\mathcal{M} = \langle W, R_1, \ldots, R_n, V\rangle$ where each $i = 1, \ldots, n$, $R_i \subseteq W \times W$

  $\mathcal{M}, w \models \Box_i\varphi$ iff for all $v \in W$, if wR_iv, then $\mathcal{M}, v \models \varphi$.

- Converse modality: $\mathcal{M} = \langle W, R, V\rangle$

  $\mathcal{M}, w \models \Box^{\leftarrow}\varphi$ iff for all $v \in W$, if vRw, then $\mathcal{M}, v \models \varphi$.

Language Extensions

- Universal modality: $\mathcal{M} = \langle W, R, V\rangle$

  $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$.

- Difference modality: $\mathcal{M} = \langle W, R, V\rangle$

  $\mathcal{M}, w \models D\varphi$ iff for all $v \in W$, if $w \neq v$, then $\mathcal{M}, v \models \varphi$.

Language Extensions

- Reflexive Transitive Closure: $\mathcal{M} = \langle W, R, V \rangle$

  $\mathcal{M}, w \models \Box^* \varphi$ iff for all $v \in W$, if wR^*v, then $\mathcal{M}, v \models \varphi$, where R^* is the reflexive transitive closure of R

- Common knowledge/belief: $\mathcal{M} = \langle W, R_1, \ldots, R_n, V \rangle$

  $\mathcal{M}, w \models \Box^* \varphi$ iff for all $v \in W$, if $w(\bigcup_i R_i)^* v$, then $\mathcal{M}, v \models \varphi$

- Distributed knowledge/belief: $\mathcal{M} = \langle W, R_1, \ldots, R_n, V \rangle$

  $\mathcal{M}, w \models [\cap] \varphi$ iff for all $v \in W$, if $wR^\cap v$, then $\mathcal{M}, v \models \varphi$, where $R^\cap = \bigcap_i R_i$

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- $V : \mathrm{At} \times W \rightarrow \{F, T\}$

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- $V : \mathrm{At} \times W \to \{F, T\}$

- Each possible world is a total function assigning truth values to atomic propositions
- Bivalent: two truth values
- Different possible worlds may be associated with the same propositional valuation

Many-valued modal logic

Truth Values: Suppose that $\mathcal{T}$ is a finite lattice. Its members are referred to as **truth values.** The lattice ordering is denoted $\leq$, and the **meet** and **join** operations by $\wedge$ and $\vee$. The bottom and the top of $\mathcal{T}$ are denoted F and T respectively, and it is assumed that $F \neq T$.

Language: The propositional language $\mathcal{L}_0^{\mathcal{T}}$ includes atomic propositions (including truth value constants for each element of $\mathcal{T}$); and is closed under conjunction ($\wedge$), disjunction ($\vee$) and implication ($\rightarrow$).

Valuation: A valuation is a mapping from the atomic formulas of $\mathcal{L}_0^{\mathcal{T}}$ to $\mathcal{T}$ that maps each member of $\mathcal{T}$ to itself.

A valuation v is extended to all formulas in the obvious way:

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- An element $c \in \mathcal{T}$ is the **pseudo-complement** of a relative to b if c is the greatest member of $\mathcal{T}$ such that $a \wedge c \leq b$. If the pseudo-complement of a relative to b exists, it is denoted by $a \Rightarrow b$; Then, $v(\varphi \to \psi) = v(\varphi) \Rightarrow v(\psi)$

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- $V(\neg\varphi) = V(\varphi) \Rightarrow F$

A **binary modal model** is a structure $\langle W, R, v \rangle$ where $W \neq \emptyset$, $R \subseteq W \times W$ and $V : (\mathrm{At} \cup \mathcal{T}) \to \mathcal{T}$ such that for $t \in \mathcal{T}$ and $w \in W$, $V(w, t) = t$.

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$$V(w, \Box\varphi) = \bigwedge\{V(v, \varphi) \mid v \in R(w)\}$$

$$V(w, \Diamond\varphi) = \bigvee\{V(v, \varphi) \mid v \in R(w)\}$$

Binary Necessitation Rule

Suppose that $t_1, \ldots, t_n, t \in \mathcal{T}$ and $\varphi_1, \ldots, \varphi_n, \psi$ are formulas

$$\frac{t_1 \to \varphi_1, \ldots, t_n \to \varphi_n \;\Rightarrow\; t \to \psi}{t_1 \to \Box\varphi_1, \ldots, t_n \to \Box\varphi_n \;\Rightarrow\; t \to \Box\psi}$$

A $\mathcal{T}$**-modal model** is a structure $\langle W, R, V\rangle$, where $W \neq \emptyset$, $R : W \times W \to \mathcal{T}$ and $V : (\mathrm{At} \cup \mathcal{T}) \to \mathcal{T}$ such that for $t \in \mathcal{T}$ and $w \in W$, $V(w, t) = t$.

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$$V(w, \Box\varphi) = \bigwedge\{R(w, w') \Rightarrow V(w', \varphi) \mid w' \in W\}$$

$$V(w, \Diamond\varphi) = \bigvee\{R(w, w') \wedge V(w', \varphi) \mid w' \in W\}$$

A formula φ is valid in a $\mathcal{T}$-modal model $\langle W, R, V\rangle$ if $V(w,\varphi) = T$ for all $w \in W$.

- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is $\mathcal{T}$-valid.
- $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ is not valid
- $(t \to \Box\varphi) \leftrightarrow \Box(t \to \varphi)$ is $\mathcal{T}$-valid
- $(\Diamond\varphi \to t) \leftrightarrow \Box(\varphi \to t)$ is $\mathcal{T}$-valid

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- $(\Diamond\varphi \to t) \leftrightarrow \Box(\varphi \to t)$ is $\mathcal{T}$-valid

Note that when $t = F$, the last item means that $\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$ and $\neg\Diamond\neg\varphi \leftrightarrow \Box\neg\neg\varphi$ are valid.

M. Fitting. *Many-valued modal logics*. Fundamenta Informaticae , 15:235 - 254, 1991.

M. Fitting. *Many-valued modal logics, II*. Fundamenta Informaticae , 17:55 - 73, 1992.

M. Fitting. *Bisimulations and Boolean Vectors*. in Advances in Modal Logic 4, pp. 97 - 125, King's College Publications, 2003.

From worlds to possibilities

I. L. Humberstone. *From Worlds to Possibilities*. Journal of Philosophical Logic, 10(3), pp. 313 - 339, 1981.

Temporal Logic: Instants vs. Intervals

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“...instants are limits of nested sequences of subintervals, and talk which takes them too seriously is diagnosed as involving what has been called the ‘infinitieth term’ fallacy. The only temporal entities robust enough for talk of truth with respect to them to be primitively intelligible...are intervals, and the salient fact about interval sub-division is that it is a process which does not terminate.”

Possibilities

Goal: develop a semantics for modal logic...

"...in which less determinate entities than possible worlds, things which I am inclined for want of a better word to call simply *possibilities*, are what sentences (or fomulae) are true or false with respect to.

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"...in which less determinate entities than possible worlds, things which I am inclined for want of a better word to call simply *possibilities*, are what sentences (or fomulae) are true or false with respect to. Our entrée into the realm of the possible, one might say, is though imagining and story-telling about how things might be, considering what might happen under hypothetical circumstances, entertaining a counterfactual supposition, and so on; but each such introduction into that realm presents us what I called...a region of logical space, and not with a point thereof."

D Edgington. *The Paradox of Knowability*. Mind, 94(376), pp. 557-568, 1985.

The Knowability Paradox

1. If 'p' is true, then it is logically possible that it is known (by someone at some time) that p.

   $\varphi \rightarrow \Diamond K\varphi$

2. There is at least one truth which is never known.

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7. $(p \wedge \neg Kp) \to \Diamond(Kp \wedge \neg Kp)$
8. $(p \wedge \neg Kp) \to \bot$

"I shall argue that an analogous solution can be given to the original paradox, in terms of the modal operator 'actually'....I shall show that the solution is consistent, makes philosophical sense, and does not violate verificationist scruple"

1' For any actual truth, it is possible to know that it is actually true

2' There is something which is *actually* true and not known to be true

" Three assumptions are needed. If the verificationist can accept these assumptions, he can avail himself of this way out of the paradox."

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"...possible worlds are far too idealized to figure in our ordinary modal talk. When I think of the possibility that I will finish the paper today, I am not thinking of one totally specific possible world. It is not the sort of thing I am capable of thinking of. It, itself, seems to violate the principle of knowability. Nor am I thinking of a large class of possible worlds in which I finish the paper. I am thinking of a possibility or a possible situation, which I can refine, or subdivide, into more specific possible situations if I wish, but which will never reach total specificity."

I. Rumfitt. *On A Neglected Path to Intuitionism*. Topoi, 31(1), pp. 101 - 109, 2012.

Intuitionistic Logic

The Law of Exclusive Middle, $\varphi \vee \neg\varphi$, is not intuitionisitically valid.

"According to W. V. Quine, in any disagreement over basic logical laws the contesting parties mean different things by the connectives or quantifiers implicated in those laws. 'Whoever denies the law of excluded middle changes the subject? In repudiating '*p* or $\sim$ *p*' he is [...] giving up classical negation, or perhaps alternation, or both' (Quine 1986, 83)...In this paper, I shall refute Quine by showing that classical and intuitionist logicians need not attach different senses to the connectives."

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That theory represents a possibility, such as my being in Pisa today, as a set of possible worlds—fully determinate ways in which the entire cosmos could have been. The notion of full determinacy needs explanation, but possible-worlds theorists give it cash value by postulating that, for any statement and any world, the statement is either true at that world or false there.

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That theory represents a possibility, such as my being in Pisa today, as a set of possible worlds—fully determinate ways in which the entire cosmos could have been. The notion of full determinacy needs explanation, but possible-worlds theorists give it cash value by postulating that, for any statement and any world, the statement is either true at that world or false there. That is, they give it content precisely by postulating the necessary truth of Bivalence. In the present dialectical context, that cannot be assumed, so our theory of consequence cannot invoke possible worlds.”

W. Holliday. *Possibility Frames and Forcing for Modal Logic*. Technical report, 2018.

Posets and Relations

A poset $\langle S, \sqsubseteq \rangle$ where S is a set and $\sqsubseteq$ is a reflexive and transitive relation on S.

For $x, y \in S$:

1. $\downarrow x = \{y \in S \mid y \sqsubseteq x\}$
2. $x \between y$ iff $\exists z \in S : z \sqsubseteq x$ and $z \sqsubseteq y$
   "x and y are compatible"
3. $x \perp y$ iff not $x \between y$
   "x and y are incompatible"

Posets and Relations

For a binary relation $R \subseteq S \times S$ and $X \subseteq S$, and $x \in S$:

1. $R[X]$ is the image of X under R, i.e.,
   $R[X] = \{y \in S \mid \exists x \in X : xRy\}$
2. $R^{-1}[X]$ is the preimage of X under R, i.e.,
   $R^{-1}[X] = \{y \in S \mid \exists x \in X : yRx\}$
3. $R(x) = R[\{x\}]$

A **partial-state frame** is a tuple $\mathcal{F} = \langle S, R, \sqsubseteq, P\rangle$ where

1. S is a nonempty set (the set of states)
2. $\sqsubseteq$ is a partial order on S (the **refinement** relation)
3. R is a binary relation on S (the accessibility relation — possibly more than one)
4. P is a subset of $\wp(S)$ such that $\emptyset \in P$ and for all $X, Y \in P$:
    4.1 $X \cap Y \in P$
    4.2 $X \supset Y = \{s \in S \mid \forall s' \sqsubseteq s : s' \in X \Rightarrow s' \in Y\} \in\in P$
    4.3 $\blacksquare Y = \{s \in S \mid R(s) \subseteq Y\} \in P$

A model is a tuple $\langle \mathcal{F}, \pi\rangle$ where $\pi : \mathrm{At} \to P$.

$x \sqsubseteq y$ means that the state x is a *refinement* or *further specification* or *extension* of the state y

Suppose that $\mathcal{M} = \langle S, R, \sqsubseteq, P, \pi \rangle$ is a partial-state model with $x \in S$:

- $\mathcal{M}, x \models p$ iff $x \in \pi(p)$
- $\mathcal{M}, x \models \neg\varphi$ iff $\forall x' \sqsubseteq x$, $\mathcal{M}, x' \not\models \varphi$
- $\mathcal{M}, x \models \varphi \wedge \psi$ iff $\mathcal{M}, x \models \varphi$ and $\mathcal{M}, x \models \psi$
- $\mathcal{M}, x \models \varphi \rightarrow \psi$ iff $\forall x' \sqsubseteq x$, if $\mathcal{M}, x' \models \varphi$ then $\mathcal{M}, x' \models \psi$
- $\mathcal{M}, x \models \Box\varphi$ iff $\forall y \in R(x)$, $\mathcal{M}, y \models \varphi$

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Fact: Given $\varphi \vee \psi \;:=\; \neg(\neg\varphi \wedge \neg\psi)$,

$\mathcal{M}, x \models \varphi \vee \psi$ iff $\forall x' \sqsubseteq x\ \exists x'' \sqsubseteq x'$, $\mathcal{M}, x'' \models \varphi$ or $\mathcal{M}, x'' \models \psi$

Suppose that $\mathcal{F} = \langle S, R, \sqsubseteq, P\rangle$ is a partial-state frame and $\mathcal{M} = \langle \mathcal{F}, \pi\rangle$ a partial-state model:

1. $[\![p]\!]_{\mathcal{M}} = \pi(p)$
2. $[\![\neg\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \supset [\![\emptyset]\!]_{\mathcal{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$
4. $[\![\varphi \to \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \supset [\![\psi]\!]_{\mathcal{M}}$
5. $[\![\Box\varphi]\!]_{\mathcal{M}} = \blacksquare[\![\varphi]\!]_{\mathcal{M}}$

Suppose that $\mathcal{F} = \langle S, R, \sqsubseteq, P\rangle$ is a partial-state frame and $\mathcal{M} = \langle \mathcal{F}, \pi\rangle$ a partial-state model:

1. $[\![p]\!]_{\mathcal{M}} = \pi(p)$
2. $[\![\neg\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \supset [\![\emptyset]\!]_{\mathcal{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$
4. $[\![\varphi \rightarrow \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \supset [\![\psi]\!]_{\mathcal{M}}$
5. $[\![\Box\varphi]\!]_{\mathcal{M}} = \blacksquare[\![\varphi]\!]_{\mathcal{M}}$

- For any formula $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathcal{M}} \in P$
- The set of formulas valid over $\mathcal{F}$ is closed under uniform substitution

World Frames

A relational frame $\langle W, R \rangle$ can be regarded as a partial-frame $\langle W, R, \sqsubseteq, P \rangle$ where

1. $\sqsubseteq$ is the identity relation
2. $P = \wp(W)$

A general relational frame $\langle W, R, A \rangle$ can be regarded as a partial-frame $\langle W, R, \sqsubseteq, A \rangle$ where

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Fact. The definition of truth for Boolean connectives reduces to the standard definition on world frames.

Intuitionistic Modal Frames

A **full intuitionistic modal frame** is a partial-state frame $\mathcal{F} = \langle S, R, \sqsubseteq, P \rangle$ satisfying:

1. up-R: if $x' \sqsubseteq x$ and $x'Ry'$, then xRy'
2. R-down: if $y' \sqsubseteq y$ and xRy, then xRy'
3. P is the set of all downsets in $\langle S, \sqsubseteq \rangle$

Persistence: If $\mathcal{M}, x \models \varphi$ and $x' \sqsubseteq x$, then $\mathcal{M}, x' \models \varphi$.

up-R

If $x' \sqsubseteq x$ and $x' R y'$, then xRy'.

$$\begin{tikzcd}
x \arrow[d] & \\
x' & y'
\end{tikzcd}$$

up-R

If $x' \sqsubseteq x$ and $x' R y'$, then xRy'.

$$\begin{tikzcd} x \arrow[d] & \\ x' \arrow[r, dashed, "R"] & y' \end{tikzcd}$$

up-R

If $x' \sqsubseteq x$ and $x' R y'$, then $x R y'$.

$$\begin{tikzcd} x \arrow[d] \arrow[dr, dashed, "R"] & \\ x' \arrow[r, dashed, "R"] & y' \end{tikzcd}$$

R-down

If $y' \sqsubseteq y$ and xRy, then xRy'

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R-down

If $y' \sqsubseteq y$ and xRy, then xRy'

$$\begin{tikzcd} x \arrow[r, dashed, "R"] \arrow[dr, dashed, "R"'] & y \arrow[d] \\ & y' \end{tikzcd}$$

Since truth sets of all formulas are downsets, we have that $\varphi \to (\psi \to \varphi)$ is intuitionistically valid.

Intuitionistic Disjunction:

$\mathcal{M}, x \models \varphi \vee \psi$ iff $\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi$

Powerset Possibilization

Given a world frame $\mathfrak{F} = \langle W, R, A \rangle$ and a world model $\mathcal{M} = \langle \mathfrak{F}, V \rangle$, the powerset possibilization are $\mathfrak{F}^{\wp} = \langle S, \sqsubseteq, R, P \rangle$ and $\mathfrak{M}^{\wp} = \langle \mathfrak{F}^{\wp}, \pi \rangle$, defined as follows:

1. $S = \wp(W) - \emptyset$
2. $X \sqsubseteq Y$ iff $X \subseteq Y$
3. XRY iff $Y \subseteq R[X]$
4. $P = \{\downarrow X \mid X \in A\}$
5. $\pi(p) = \{X \in S \mid X \subseteq V(p)\}$

Fact.

1. For any $X \in \mathfrak{M}^{\wp}$ and $\varphi \in \mathcal{L}$, $\mathfrak{M}^{\wp}, X \models \varphi$ iff $\forall x \in \mathfrak{M}$, $\mathfrak{M}, x \models \varphi$
2. For any set of formulas Σ, Σ is satisfiable over $\mathfrak{F}^{\wp}$ iff Σ is satisfiable over $\mathfrak{F}$

Corollary. **K** is sound with respect to the class of all powerset possibilizations of world frames and complete with respect to the class of powerset possibilizations of full world frames.

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Corollary. **K** is sound with respect to the class of all powerset possibilizations of world frames and complete with respect to the class of powerset possibilizations of full world frames. Moreover, any normal modal logic that is sound and complete with respect to a class $\mathbb{F}$ of world frames, according to standard Kripke semantics, is also sound and complete with respect to the class of powerset possibilizations of frames from $\mathbb{F}$, according to partial-state semantics.

Possibility frames

Note that $\varphi \leftrightarrow \neg\neg\varphi$ is not valid on partial-state frames.

$\neg\neg\varphi \to \varphi$:

Refinability: If $\mathcal{M}, x \not\models \varphi$ then there is a $x' \sqsubseteq x$ such that $\mathcal{M}, x' \models \neg\varphi$

If φ is indeterminate at x, i.e., if $\mathcal{M}, x \not\models \varphi$ and $\mathcal{M}, x \not\models \neg\varphi$, then there is a refinement of x that decides φ negatively and there is a refinement of x that decides φ affirmatively.

Indeterminacy of φ is equivalent to having refinements that decide φ each way.

$\varphi \to \neg\neg\varphi$:

Persistence: if $\mathcal{M}, x \models \varphi$ and $x' \sqsubseteq x$, then $\mathcal{M}, x' \models \varphi$.

In classical partial-state frames, every admissible proposition $X \in P$ will satisfy:

- Persistence: if $x \in X$ and $x' \sqsubseteq x$, then $x' \in X$
- Refinability: if $x \notin X$ then $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x' :\ x'' \notin X$

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In intuitionistic models, the admissible propositions are all the downsets, but in classical models, the admissible propositions are all downsets that also satisfy admissibility.

X satisfies both persistence and refinability is equivalent to X satisfying:

$x \in X$ iff $\forall x' \sqsubseteq x'\ \exists x'' \sqsubseteq x' :\ x'' \in X$

Proposition. The conditions of persistence and refinability on admissible propositions are necessary and sufficient for a partial-state frame to be classical.

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- Every state x belongs to a chain $x_0 \sqsupseteq x_1 \sqsupseteq \cdots$ that decides the truth value of every formula eventually: For all φ, there is $k \in \mathbb{N}$ such that $\mathcal{M}, x_k \models \varphi$ or $\mathcal{M}, x_k \models \neg\varphi$.

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- Every state x belongs to a chain $x_0 \sqsupseteq x_1 \sqsupseteq \cdots$ that decides the truth value of every formula eventually: For all φ, there is $k \in \mathbb{N}$ such that $\mathcal{M}, x_k \models \varphi$ or $\mathcal{M}, x_k \models \neg\varphi$.
- **Lemma.** Fix a language with countably many propositional variables. If $\mathcal{F} = \langle S, R, \sqsubseteq, P \rangle$ is a partial-state frame in which every $X \in P$ satisfies persistence and refinability, and if α is a propositional tautology, then α is valid over $\mathcal{F}$.

Let $\mathcal{O}(S, \sqsubseteq)$ be the set of all downsets in $\langle S, \sqsubseteq \rangle$.

$\langle S, \mathcal{O}(S, \sqsubseteq) \rangle$ is a topology (the downset, or Alexandrov, topology).

Interior: $int(X)$ is the largest downset included in X
Closure: $cl(X)$ is the smallest upset that includes X

$[\![\neg\varphi]\!]_{\mathcal{M}} = int(S - [\![\varphi]\!]_{\mathcal{M}})$
$[\![\varphi \rightarrow \psi]\!]_{\mathcal{M}} = int((S - [\![\varphi]\!]_{\mathcal{M}}) \cup [\![\psi]\!]_{\mathcal{M}})$
$[\![\varphi \vee \psi]\!]_{\mathcal{M}} = int(cl([\![\varphi]\!]_{\mathcal{M}} \cup [\![\psi]\!]_{\mathcal{M}}))$

A set X is **regular open** of $X = int(cl(X))$.

Fact. For any topological space $\langle S, \mathcal{O} \rangle$, the structure $\langle RO(S), \wedge, -, \top \rangle$ where

$RO(S)$ is the set of all regular open sets in the topology, $X \wedge Y = X \cap Y$, $-X = int(S - X)$, and $\top = S$

is a complete Boolean algebra with

for all $\mathcal{X} \subseteq R(S)$, $\bigwedge \mathcal{X} = int(\bigcap \mathcal{X})$ and $\bigvee \mathcal{X} = int(cl(\bigcup \mathcal{X}))$.

Lemma. For any poset $\langle S, \sqsubseteq \rangle$ and $X \subseteq S$:

1. $int(cl(X)) = \{x \in S \mid \forall x' \sqsubseteq x \; \exists x'' \sqsubseteq x' : \; x'' \in X\}$
2. $int(cl(\Downarrow X))$ is the smallest regular open set that includes X, where $\Downarrow X = \{y \in S \mid \exists x \in X : \; y \sqsubseteq x\}$
3. X satisfies persistence and refinability iff X is regular open in $\mathcal{O}(S, \sqsubseteq)$

Proposition For any partial-state frame $\mathcal{F} = \langle S, \sqsubseteq, R, P\rangle$ the following are equivalent:

1. the set of $\varphi \in \mathcal{L}$ valid over $\mathcal{F}$ is a classical normal modal logic;
2. for every $\varphi \in \mathcal{L}$, $\neg\neg\varphi$ is equivalent to φ over $\mathcal{F}$; and
3. $P \subseteq RO(\mathcal{F})$

Definition. A **possibility frame** is a partial-state frame $\mathcal{F} = \langle S, \sqsubseteq, R, P\rangle$ in which $P \subseteq RO(\mathcal{F})$. A full possibility frame is a possibility frame in which $P = RO(\mathcal{F})$

An important property of a full possibility frame $\mathcal{F}$ is that $RO(\mathcal{F})$ is closed under $\blacksquare$.

This is not trivial, for there are possibility frames $\mathcal{F}$ that lack the property.

By contrast, it is easy to check that for any $\mathcal{F}$, $RO(\mathcal{F})$ is closed under $\cap$ and $\supset$.

The fact that not every possibility frame is such that $RO(\mathcal{F})$ is closed under $\blacksquare$ means that not every possibility frame can be turned into a full possibility frame simply by replacing its set of admissible propositions P by $RO(\mathcal{F})$.

For any poset $\langle S, \sqsubseteq \rangle$ and binary relation R on S the following are equivalent:

1. $RO(S, \sqsubseteq)$ is closed under $\blacksquare$
2. R and $\sqsubseteq$ and satisfy:
   2.1 R-rule: if $x' \sqsubseteq x$ and $x'\ R\ y' \between z$, then $\exists y:\ x\ R\ y \between z$
   2.2 R$\Rightarrow$win: if xRy, then
   $\forall y' \sqsubseteq y\ \exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x' \exists y'' \between y':\ x''Ry''$

- If $\mathcal{F}$ satisfies R-rule, then $\mathcal{F}$ satisfies up-R
- If $\mathcal{F}$ satisfies down-R, then R$\Rightarrow$win is equivalent with:
  R-refinability: if xRy then $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x'\ \exists y' \sqsubseteq y:\ x''Ry'$

if xRy then $\exists x' \sqsubseteq x \; \forall x'' \sqsubseteq x' \; \exists y' \sqsubseteq y : \; x'' R y'$.

$$
\begin{tikzcd}
x'' & \\
x' & y' \\
x \arrow[r, dashed, "R"'] & y
\end{tikzcd}
$$

if xRy then $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x'\ \exists y' \sqsubseteq y :\ x'' R y'$.

$$
\begin{tikzcd}
x'' & \\
x' & y' \\
x \arrow[u, "\exists"] \arrow[r, dashed, "R"'] & y
\end{tikzcd}
$$

if xRy then $\exists x' \sqsubseteq x \; \forall x'' \sqsubseteq x' \; \exists y' \sqsubseteq y : \; x'' R y'$.

$$
\begin{tikzcd}
x'' \arrow[dr, "R"] & \\
x' \arrow[u, "\forall"] & y' \\
x \arrow[u, "\exists"] \arrow[r, dashed, "R"'] & y \arrow[u, "\exists"]
\end{tikzcd}
$$

The are full possibility frames $\mathcal{F}$ that validate a modal formula that is not valid on any Kripke frame. Thus, the logic of $\mathcal{F}$ will be a normal modal logic that is Kripke-frame inconsistent—it is not sound with respect to any Kripke frame.

Suppose that φ and ψ for formulas such that the propositional variable p does not occur in ψ. The consider the following formula:

$$\text{SPLIT} \qquad \Diamond_i(p \wedge \psi) \to (\Diamond_i(p \wedge \varphi) \wedge \Diamond_i(p \wedge \neg\varphi))$$

Any Kripke frame $\mathfrak{F}$ that validates SPLIT must also validate $\neg\Diamond_i\psi$.

Worlds cannot split, but possibilities can: There is a full possibility frames that validates and instance of SPLIT and $\Diamond_i\psi$.

There are three approaches to valuation functions in the literature on possibility semantics.

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The approach followed here: a valuation is a total function $\pi : \mathsf{At} \to \wp(S)$ such that $\pi(p)$ satisfies persistence and refinability.

$x \in \pi(p)$ means that x determines that p si true and $x \notin \pi(p)$ means that x does not determine that p is true, i.e., that either x determines that p is false or x does not determine the truth or falsity of p.

Partial Valuations

A valuation is a partial function $V : \mathrm{At} \times S \to \{0, 1\}$ satisfying stability and resolution:

1. stability $V(p, x)$ is defined and $x' \sqsubseteq x$, then $V(p, x')$ is defined and $V(p, x) = V(p, x')$
2. resolution: if $V(p, x)$ is undefined, then there are $y \sqsubseteq x$ and $z \sqsubseteq x$ such that $V(p, y) = 1$ and $V(p, z) = 0$.

$V(p, x) = 1$ means that x determines that p is true; $V(p, x) = 0$ means that x determines that p is false; $V(p, x)$ being undefined means that x does not determine the truth or falsity of p.

Total Valuations

$U : \mathrm{At} \times S \to 0, 1$ is a total function such that $\{x \in S \mid U(p, x) = 1\}$ satisfies persistence and refinability in the sense of this paper; $U(p, x) = 1$ means that x determines that p is true; $U(p, x) = 0$ means that x does not determine that p is true.

M. Harrison-Trainor. *Worldization of Possibility Models*. manuscript, 2018.

From possibilities to worlds

"[T]he business of making a possibility more determinate seems openended. There are possibilities that the child at home should be a boy, a six-year-old boy, a six-year-old boy with blue eyes, a six-year old boy with blue eyes who weighs 3 stone, and so forth. So far from terminating in a fully determinate possibility, we seem to have an indefinitely long sequence of increasingly determinate possibilities, any one of which is open to further determination. But then, so far from conceiving of our rational activities as discriminating between regions of determinate points, we appear to have no clear conception of such a point at all. "

I. Rumfitt. *The Boundary Stones of Thought: An Essay in the Philosophy of Logic*. Oxford University Press, 2015.

Theorem (Harrison-Trainor). Let $\mathcal{M}$ be a countable possibility model in a countable language. Then there is a Kripke model $\mathcal{K}$ which is a **worldization** of $\mathcal{M}$.

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Note: there are counterexamples if $\mathcal{M}$ is not countable or the language is not countable.

J. van Benthem, N. Bezhanishvili and W. Holliday. *A Bimodal Perspective on Possibility Semantics*. Journal of Logic and Computation, 27(5), 2017, pp. 1353 - 1389.