Introduction to First Order Modal Logic

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"[W]hat is first-order modal logic for? What do quantifiers add to the mix? Motivations based on natural language and philosophy are still central, though we have a much richer variety of things we can potentially formalize and investigate. Of course we want a semantics that agrees with our intuitive understanding, but now intuitions can, and do, differ substantially from person to person. Are designators rigid? Can objects exist in more than one possible world? Should there be a distinction between identity and necessary identity? And for that matter, is the whole subject a mistake from the beginning, as Quine would have it? Rather than a semantics on which we all generally agree, quite a disparate range has been proposed. We are still exploring what first-order modal semantics should be; the propositional case was settled long ago."

(Fitting, pg. 1, First Order Intensional Logic)

First Order Modal Language

Let $\mathcal{V}$ be a set of variables and $\mathcal{C}$ a set of constants.

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Let *Pred* be a set of predicate symbols. A **formula** is constructed by any

$$\varphi \quad := \quad t_1 = t_2 \mid P(t_1, \ldots, t_n) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi \mid (\forall x)\varphi \mid (\exists x)\varphi$$

where $P \in Pred$ of arity n, $t_i \in \mathcal{T}$ for $i = 1, \ldots, n$ and $x \in \mathcal{V}$

(Sometimes equality is not in the language)

- $\forall x \Box P(x)$ $\exists x \Diamond P(x)$
- $\Box \forall x P(x)$ $\Diamond \exists x P(x)$

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- $\Box \forall x \varphi(x) \rightarrow \forall x \Box \varphi(x)$
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- $\Box \forall x \varphi(x) \rightarrow \forall x \Box \varphi(x)$
- $\forall x \Box \varphi(x) \rightarrow \Box \forall x \varphi(x)$

- $(x = y) \rightarrow \Box(x = y)$
- $(x = c) \rightarrow \Box(x = c)$
- $(x \neq y) \rightarrow \Box(x \neq y)$

Overviews

T. Braüner and S. Ghilardi. *First-order Modal Logic*. Handbook of Modal Logic, pgs. 549 - 620 (2007).

D. Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic*. Elsevier, 2009.

M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers (1998).

Constant vs. Varying Domains

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where $W \neq \emptyset$ and D are sets, and $R \subseteq W \times W$.

A **varying domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W is a non-empty set, $R \subseteq W \times W$, and for each $w \in W$, $D(w)$ is a set (the domain at w). Let the **domain of the model** be $D = \bigcup_{w\in W} D(w)$.

Substitutions

Suppose that D is the domain of the model.

A **substitution** is any function $s : \mathcal{V} \to D$ ($\mathcal{V}$ the set of variables).

A substitution s' is said to be an x-**variant** of s, denoted $s \sim_x s'$, if for all $y \in \mathcal{V}$, if $y \neq x$, then $s(y) = s'(y)$.

First Order Interpretations

Let D be the domain.

An **interpretation** I assigns an n-ary relation to each n-ary predicate symbol and an element of the domain to each constant symbol:

If P is an n-ary predicate symbol, then $I(P) \subseteq D^n$

If c is constant, then $I(c) \in D$

If $t \in \mathcal{T}$ is a term, I is an interpretation and s is a substitution, then $t^{I,s} \in D$, where $t^{I,s}$ is $I(t)$ if $t \in \mathcal{C}$ and $t^{I,s}$ is $s(t)$ if $t \in \mathcal{V}$

Interpretation in a Kripke Model

Let D be the domain for a Kripke model with worlds W.

An **interpretation** I assigns an n-ary relation to each n-ary predicate symbol and world w and an element of the domain to each constant symbol and world w:

If P is an n-ary predicate symbol, then $I(P, w) \subseteq D^n$

If c is constant, then $I(c, w) \in D$

If $t \in \mathcal{T}$ is a term, I is an interpretation and s is a substitution and $w \in W$, then $t^{I,s,w} \in D$, where $t^{I,s,w}$ is $I(t, w)$ if $t \in \mathcal{C}$ and $t^{I,s}$ is $s(t)$ if $t \in \mathcal{V}$

Truth

Let $\mathcal{M} = \langle W, R, D, I \rangle$ be a (varying/constant) domain Kripke model:

- $\mathcal{M}, w \models_s t_1 = t_2$ iff $t_1^{I,s,w} = t_2^{I,s,w}$
- $\mathcal{M}, w \models_s P(t_1, \ldots, t_n)$ iff $\langle t_1^{I,s,w}, \ldots, t_n^{I,s,w} \rangle \in I(P, w)$
- $\mathcal{M}, w \models_s \neg\varphi$ iff $\mathcal{M}, w \not\models_s \varphi$
- $\mathcal{M}, w \models_s \varphi \wedge \psi$ iff $\mathcal{M}, w \models_s \varphi$ and $\mathcal{M}, w \models_s \psi$

Varying Domains

Let $\mathcal{M} = \langle W, R, D, I \rangle$ be a varying domain Kripke model:

- $\mathcal{M}, w \models_s \Box\varphi$ iff for all $v \in W$, if wRv, then $\mathcal{M}, v \models_s \varphi$
- $\mathcal{M}, w \models_s \forall x\varphi$ iff for all s', if $s \sim_x s'$ and $s'(w) \in D(w)$, then $\mathcal{M}, w \models_{s'} \varphi$

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- *Actualist* quantification: only quantifying over objects that exist
- $\forall x P(x) \rightarrow P(y)$ is *not* valid (cf. *Free logic*)
- Can add possibilist quantifiers

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- *Actualist* quantification: only quantifying over objects that exist
- $\forall x P(x) \rightarrow P(y)$ is *not* valid (cf. *Free logic*)
- Can add possibilist quantifiers
- We can say "y exists": $\exists x(x = y)$,
  "y doesn't exists": $\neg\exists x(x = y)$,
  but we cannot express "there are non-existents"

Barcan Schemas

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- **Barcan formula** (BF): $\forall x \Box \varphi(x) \rightarrow \Box \forall x \varphi(x)$
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Lemma. *CBF* is valid in a varying domain relational frame iff the frame is monotonic.

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Lemma. BF is valid in a varying domain relational frame iff the frame is anti-monotonic

A varying domain is **anti-monotonic** if for all $w, v \in W$, if wRv, then $D(v) \subseteq D(w)$

Constant Domain Models

Let $\mathcal{M} = \langle W, R, D, I \rangle$ be a constant domain Kripke model:

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- *Possibilist* quantification: quantifying over all objects (even non-existent objects)
- $\forall xP(x) \rightarrow P(y)$ is valid
- Can add actualist quantifiers:
  - Introduce an **existence predicate** E (typically assume $I(E, w) \neq \emptyset$ for all $w \in W$ and $\bigcup_w I(E, w) = D$)
  - $\forall^E x\varphi \ := \ \forall x(E(x) \rightarrow \varphi)$
  - $\exists^E x\varphi \ := \ \exists x(E(x) \wedge \varphi)$

- Since varying domain semantics can be simulated using constant domain semantics and relativized quantifiers, from a semantic point of view there is really little point in studying the varying domain version in much detail.

- Axiomatic systems intended for constant domain systems have more complex completeness proofs.

- Prefixed tableau systems for constant domain systems are considerably simpler than the varying domain versions.

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- $(x = y) \rightarrow \Box(x = y)$ is valid
- $(x \neq y) \rightarrow \Box(x \neq y)$ is valid
- How should we interpret $\Diamond P(c)$? Two possibilities:
  - The current interpretation of c has the "Possible-P" property
  - there is a possible world such that c (interpreted in that possible world) has the property P

M. Fitting. *Intensional Logic*. Stanford Encyclopedia of Philosophy, 2006. Substantive revision 2015.

M. Fitting. *First-order intensional logic*. Annals of Pure and Applied Logic, 127: 171–193, 2004.

Lewis Counterpart Semantics

A counterpart relation on a set D is a binary relation C whose *domain* and *codomain* is D.

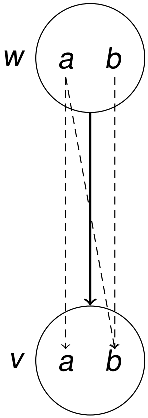
If s and s' are two valuations in D and C is a counterpart relation on D, say s' is a C-counterpart to s provided, for each variable x, $\langle s(x), s'(x)\rangle \in C$

Lewis Counterpart Semantics

A **Lewis counterpart model** is a structure $\mathcal{M} = W, R, D, C, I\rangle$ where everything is as before, except C maps each member of $W \times W$ to a counterpart relation on D.

The idea is that if $\langle d, c\rangle \in C(w, v)$, then c is a counterpart in world v of the object d in world w.

- $\mathcal{M}, w \models_s \Box\varphi$ iff $\mathcal{M}, v \models_{s'} \varphi$ for all $v \in R(w)$ and every valuation s' that is a $C(w, v)$ counterpart of s
- $\mathcal{M}, w \models_s \Diamond\varphi$ iff $\mathcal{M}, v \models_{s'} \varphi$ for some $v \in R(w)$ and some valuation s' that is a $C(w, v)$ counterpart of s

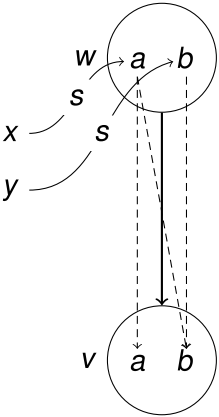


$I(P, w) = \emptyset$

$I(P, w) = \{(a, b), (b, b)\}$

$\mathcal{M}, v \models_s P(x, y) \quad \mathcal{M}, v \models_{s'} P(x, y)$

Suppose s, s' are substitutions where:
$s(x) = a$, $s(y) = b$ and $s'(x) = b$, $s'(y) = b$



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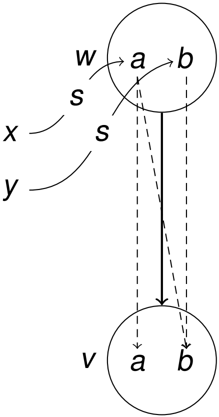

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Suppose s, s' are substitutions where:
$s(x) = a$, $s(y) = b$ and $s'(x) = b$, $s'(y) = b$

“In counterpart semantics, objects are present since they are what counterpart relations connect, but the counterpart network is fundamental, and an object, at a world, is actually something like a slice across that network. The morning star/evening star object in this world has, in an alternative Babylonian world, two counterparts, one playing the morning star role, the other the evening star role....

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First Order Intensional Logic

In addition to objects there will be what we call *intensions* or *intensional objects* or *concepts*.

Typical informal intensions are *the morning star*, *the oldest person in the world*, or simply *that*.

Intensions designate different objects under different circumstances—they are non-rigid designators.

They will be modeled by functions from possible worlds to objects. There will be quantification over intensions, as well as quantification over objects.

An intension f picks out an object at each world.

Given a unary predicate P, $P(f)$ could mean the intension f has the property P or the object designated by f has the property P. (Both make sense.)

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De Re/De Dicto issues:

- $P(f)$ is true at w if the object picked out by f at w has property P
- What about $\Diamond P(f)$?
  - (de re) $\Diamond P(f)$ is true at w if the object picked out by f at w has the property P at an accessible world v
  - (de dicto) $\Diamond P(f)$ is true at w if there is an accessible world v such that the object picked out by f at v has the property P

Predicate Abstraction

- (de re) $\Diamond P(f)$ is true at w if the object picked out by f at w has the property P at an accessible world v.

  $\langle \lambda x.\Diamond P(x)\rangle(f)$

- (de dicto) $\Diamond P(f)$ is true at w if there is an accessible world v such that the object picked out by f at v has the property P.

  $\Diamond\langle \lambda x.P(x)\rangle(f)$

Lambda Notation

Describing Functions:

- $f : \mathbb{R} \to \mathbb{R}$, where for all $x \in \mathbb{R}$, $f(x) = x^2$
- $x \mapsto x^2$
- $\lambda x.x^2$

Beta Reduction:

- $f(3) = 9$
- $(\lambda x.x^2)(3) = 3^2$

Propositional modal logic:

$$[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \to \wp(W)$$

For each formula of first-order modal logic φ:

$$[\![\varphi]\!]_{\mathcal{M}} : D^{\mathcal{V}} \to \wp(W)$$

Suppose that the possible worlds are people, and f is the *favorite-book* concept picking out, for each person, that person's favorite book. And suppose P is intended to be the *is-an-important-concept* predicate.

For a person who considers reading important, P(f) will most likely be true—the concept of a favorite book would be important for that person.

Let us say Q is intended to be the *is-an-important-book* predicate.

I certainly think $\langle \lambda x.Q(x)\rangle(f)$ is true—for me it says my favorite book is an important book (for me).

I would not think $\langle \lambda x.\Box Q(x)\rangle(f)$ to be true—for me it says that my favorite book is an important book for everybody.

On the other hand I probably would think that $\Box\langle \lambda x.Q(x)\rangle(f)$ is true—for me it says that everybody thinks their favorite book is important.

The King of Sweden could be taller than he is now.

m is an intensional variable selecting the monarch in a world.

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$\langle \lambda y.\Diamond\langle \lambda x.T(x, y)\rangle(m)\rangle(m)$

A FOIL model is a structure $\mathcal{M} = \langle W, R, D_O, D_I, I \rangle$, where $W \neq \emptyset$, $R \subseteq W \times W$, D_O is a non-empty set of objects, and D_I is a non-empty set of functions from W to D_O. Finally, I is an interpretation assigning to each predicate symbol P a relation of an appropriate type.

$\mathcal{M}, w \models_s \langle \lambda x.\varphi \rangle(f)$ iff $\mathcal{M}, w \models_{s'} \varphi$ where for all $y \in \mathcal{V}$, if $y \neq x$, then $s'(y) = s(y)$ and $s'(x) = s(f)(w)$.

Valid:

$$\forall x \forall y((x = y) \rightarrow \Box(x = y)))$$

$$\forall x \forall y((x \neq y) \rightarrow \Box(x \neq y)))$$

$$\forall f \forall g[\langle \lambda x, y.(x = y)\rangle(f, g) \rightarrow \langle \lambda x, y.\Box(x = y)\rangle(f, g)]$$

Valid:

$$\forall x \forall y ((x = y) \rightarrow \Box (x = y)))$$

$$\forall x \forall y ((x \neq y) \rightarrow \Box (x \neq y)))$$

$$\forall f \forall g [\langle \lambda x, y.(x = y) \rangle (f, g) \rightarrow \langle \lambda x, y. \Box (x = y) \rangle (f, g)]$$

Not Valid:

$$\forall f \forall g [\langle \lambda x, y.(x = y) \rangle (f, g) \rightarrow \Box \langle \lambda x, y.(x = y) \rangle (f, g)]$$

$D(f, x)$ abbreviates $\langle \lambda y.y = x \rangle(f)$ (where x and y are distinct object variables).

$\mathcal{M}, w \models_s D(f, x)$ when f designates the object x at w

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$\mathcal{M}, w \models_s D(f, x)$ when f designates the object x at w

(1) $\quad (\forall x)(\forall f)D(f, x)$

(2) $\quad (\forall x)\varphi \leftrightarrow (\forall f)\langle \lambda x.\varphi \rangle(f)$

In FOIL, (1) implies (2).

For a propositional modal logic **L**, **FOIL**-**L** is the intensional logic built on that class of frames in the obvious way. Now, let **FOIL**-**L**-λ be the restriction of **FOIL**-**L** to the sublanguage without quantifiers.

1. If **L** is one of **K**, **T**, or **D**, then **FOIL**-**L**-λ is decidable.
2. **FOIL**-**S4**-λ is undecidable, with or without equality.
3. If $=$ is interpreted by equality on D_O, then **FOIL**-**L**-λ is undecidable for any **L** between **K4** and **S5**.
4. The two preceding items remain true even if formulas are restricted to contain no object variables and only a single intension variable.
5. Tableau systems for **FOIL**-**L**-λ

M. Fitting. *Modal logics between propositional and first-order*. Journal of Logic and Computation, 12:1017 - 1026, 2002.

Constants and Function Symbols

M. Fitting. *On Height and Happiness*. in Rohit Parikh on Logic, Language and Society, Springer Outstanding Contributions to Logic, C. Baskent, L. Moss, R. Ramanujam editors, pages 235-258, 2017.

The King of Sweden could be taller than he is now.

$\langle \lambda y. \Diamond \langle \lambda x. T(x, y) \rangle (m) \rangle (m)$

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$\langle \lambda y.\Diamond\langle \lambda x.T(x,y)\rangle(m)\rangle(m)$

Alice could be taller than she is now.

$\langle \lambda y.\Diamond\langle \lambda x.T(x,y)\rangle(a)\rangle(a)$

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Problem: Names are rigid.

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So, the above two formulas imply:

$\langle \lambda x.\Diamond T(x,x)\rangle(a)$

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$\langle \lambda y.\Diamond\langle \lambda x.G(x,y)\rangle(h(a))\rangle(h(a))$

There is a family of object variables, typically x, y, ..., and intension constants, a, b, ...

We also have intension function symbols, f, g, ... of various arities, which take object variables as arguments.

Relation symbols, P, Q, ... of various arities, also taking object variables as arguments in the usual way.

An **intension function term** is $f(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are object variables and f is an n-ary intension function symbol.

Note that intension functions are not allowed to be nested—arguments are object variables

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$$P(f(g(a)))$$

is an abbreviation for

$$\langle \lambda z.\langle \lambda y.\langle \lambda x.P(x)\rangle (f(y))\rangle (g(z))\rangle (a)$$

If f is an n-ary intension function symbol then $I(f) : S \to (D^n \to D)$, for some $S \subseteq W$.

$I(f)$ is an n-ary function from D to itself from some set of possible worlds, so f may not designate at some worlds.

If a is a constant, then $I(a)$ is a 0-ary function on D, so $I(a)$ is a partial function from W to D.

If $I(f) : S \to (D^n \to D)$, we say f designates at the worlds in S.

Previous definition, where f is an intensional variable

$\mathcal{M}, w \models_s \langle \lambda y.\varphi \rangle(f)$ iff $\mathcal{M}, w \models_{s'} \varphi$ where for all $z \in \mathcal{V}$, if $z \neq y$, then $s'(z) = s(z)$ and $s'(y) = s(f)(w)$.

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Assume f is an n-ary intensional function symbol

$\mathcal{M}, w \models_s \langle \lambda y.\varphi \rangle(f(x_1, \ldots, x_n))$ iff $\mathcal{M}, w \models_{s'} \varphi$ where for all $z \in \mathcal{V}$, if $z \neq y$, then $s'(z) = s(z)$ and $s'(y) = I(f)(w)(v(x_1), \ldots, v(x_n))$.

H. Arlo Costa and E. Pacuit. *First-Order Classical Modal Logic*. Studia Logica, **84**, pgs. 171 - 210 (2006).

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

Suppose that s is a substitution.

1. $\mathcal{M}, w \models_s P(x_1, \ldots, x_n)$ iff $\langle s(x_1), \ldots, s(x_n)\rangle \in I(P, w)$
2. $\mathcal{M}, w \models_s \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]_{\mathcal{M},s}$
3. $\mathcal{M}, w \models_s \forall x\varphi$ iff for each x-variant s', $\mathcal{M}, w \models_{s'} \varphi$

First-order Modal Logic

A **constant domain Neighborhood frame** is a tuple $\langle W, N, D\rangle$ where W and D are sets, and $N : W \to \wp(\wp(W))$.

A **constant domain Neighborhood model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

Suppose that s is a substitution.

1. $\mathcal{M}, w \models_s P(x_1, \ldots, x_n)$ iff $\langle s(x_1), \ldots, s(x_n)\rangle \in I(P, w)$
2. $\mathcal{M}, w \models_s \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M},s} \in N(w)$
3. $\mathcal{M}, w \models_s \forall x\varphi$ iff for each x-variant s', $\mathcal{M}, w \models_{s'} \varphi$

Example

Suppose that F is a unary predicate symbol, $\mathcal{V} = \{x, y\}$, and $\langle W, N, D, I\rangle$ is a first order constant domain neighborhood model where

- $W = \{w, v, u\}$;
- $N(w) = \{\{w, v\}, \{u\}\}$, $N(v) = \{\{v\}\}$, $N(u) = \{\{w, v\}, \{v\}\}$;
- $D = \{a, b\}$; and
- $I(F, w) = \{a\}$, $I(F, v) = \{a, b\}$, and $I(F, u) = \emptyset$.

Example

There are four possible substitutions:

- $s_1 : \mathcal{V} \to D$ where $s_1(x) = a$, $s_1(y) = b$;
- $s_2 : \mathcal{V} \to D$ where $s_2(x) = b$, $s_2(y) = a$;
- $s_3 : \mathcal{V} \to D$ where $s_3(x) = s_3(y) = a$; and
- $s_4 : \mathcal{V} \to D$ where $s_4(x) = s_4(y) = b$

- $[\![F(x)]\!]_{\mathcal{M},s_1} = \{w, v\}$;
- $[\![F(x)]\!]_{\mathcal{M},s_2} = \{v\}$;
- $[\![F(x)]\!]_{\mathcal{M},s_3} = \{w, v\}$; and
- $[\![F(x)]\!]_{\mathcal{M},s_4} = \{v\}$.

Example

In general, every formula $\varphi \in \mathcal{L}_1$ is associated with a function

$$[\![\varphi]\!] : D^{\mathcal{V}} \rightarrow \wp(W)$$

Example

- $[\![\Box F(x)]\!]_{\mathcal{M},s_1} = [\![\Box F(x)]\!]_{\mathcal{M},s_3} = \{w, u\}$
  $[\![\Box F(x)]\!]_{\mathcal{M},s_2} = [\![\Box F(x)]\!]_{\mathcal{M},s_4} = \{v, u\}$;
- $[\![\Box \forall x F(x)]\!]_{\mathcal{M},s_1} = \{v\}$; and
- $[\![\forall x \Box F(x)]\!]_{\mathcal{M},s_1} = \{v, u\}$.

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

Barcan Schemas

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Observation 1: *CBF* is provable in **FOL** $+$ **EM**

Observation 2: *BF* and *CBF* both valid on relational frames with constant domains

Observation 3: *BF* is valid in a *varying* domain relational frame iff the frame is anti-monotonic; *CBF* is valid in a *varying* domain relational frame iff the frame is monotonic.

See (Fitting and Mendelsohn, 1998) for an extended discussion

Constant Domains without the Barcan Formula

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Constant Domains without the Barcan Formula

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Of course, *BF* should fail in this case, given that it instantiates cases of what is usually known as the '**lottery paradox**':

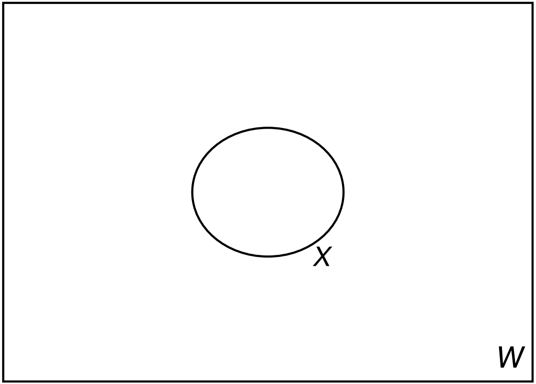
For each individual x, it is *highly probably* that x will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.

Converse Barcan Formulas and Neighborhood Frames

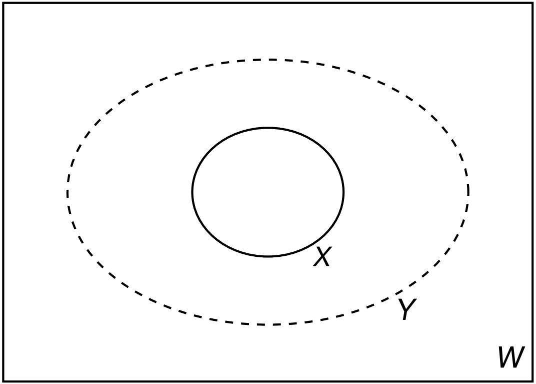
A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

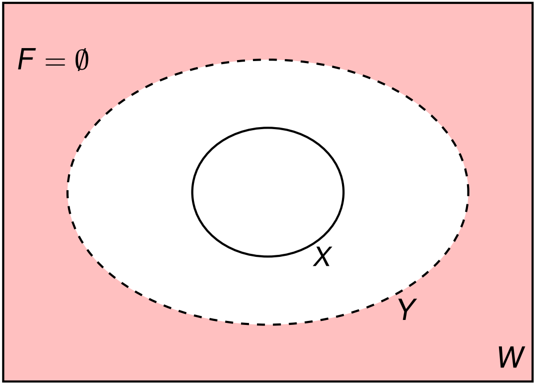
Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is monotonic.



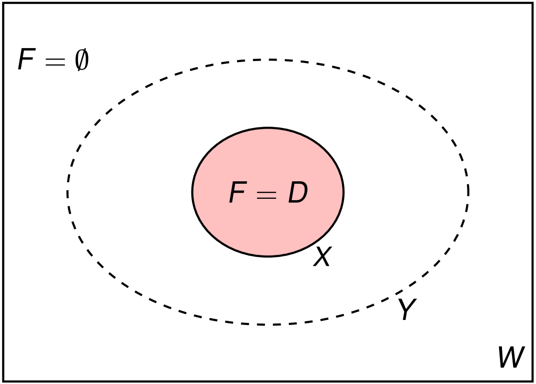
$$X \in N(w)$$



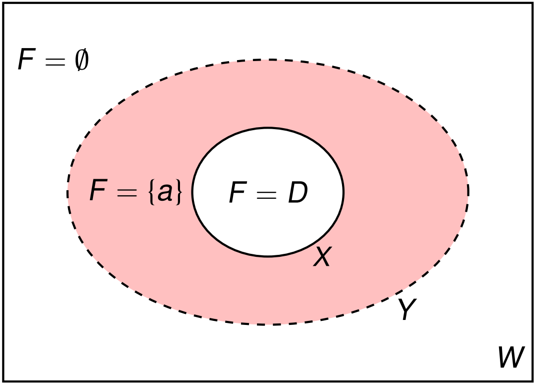
$$Y \notin N(w)$$



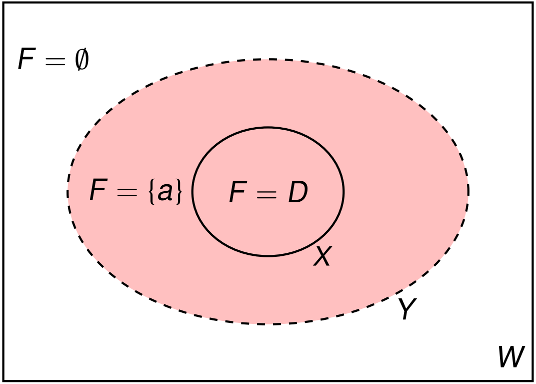
$$\forall v \notin Y, \;\; I(F, v) = \emptyset$$



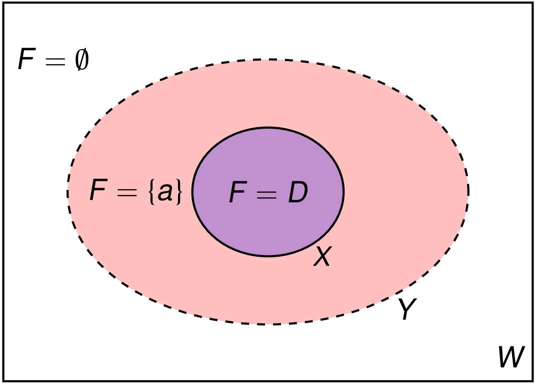
$$\forall v \in X, \;\; I(F, v) = D = \{a, b\}$$



$$\forall v \in Y - X, \;\; I(F, v) = D = \{a\}$$



$$\color{red}{(F[a])^{\mathcal{M}}} = Y \notin N(w) \quad \text{hence} \quad w \not\models \forall x \Box F(x)$$



$$(\forall x F(x))^{\mathcal{M}} = (F[a])^{\mathcal{M}} \cap (F[b])^{\mathcal{M}} = X \in N(w)$$

hence $w \models \Box \forall x F(x)$

Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i \in I} X_i \in N(w)$.

Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

Suppose that **L** is a propositional modal logic. Let **FOL** $+$ **L** denote the set of formulas closed under the following rules and axiom schemes

L All axiom schemes and rules from **L**.

(All) $\forall x\varphi(x) \rightarrow \varphi[y/x]$ is an axiom scheme, where y is free for x in φ.

(Gen) $\dfrac{\varphi \rightarrow \psi}{\varphi \rightarrow \forall x\psi}$, where x is not free in φ.

Theorem FOL $+$ E is sound and strongly complete with respect to the class of **all** constant domain neighborhood frames.

CBF

$$\vdash_{\textbf{FOL+EM}} \Box\forall x\varphi(x) \rightarrow \forall x\Box\varphi(x)$$

$$\nvdash_{\textbf{FOL+E}+(\text{CBF})} \Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$$

Completeness Theorems

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Theorem FOL $+$ **E** $+$ *CBF* is sound and strongly complete with respect to the class of frames that are either non-trivial and monotonic or trivial and not monotonic.

FOL + **K** and **FOL** + **K** + *BF*

Theorem **FOL** + **K** is sound and strongly complete with respect to the class of filters.

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Observation The augmentation of the smallest canonical model for **FOL** + **K** is not a canonical model for **FOL** + **K**. In fact, the closure under infinite intersection of the minimal canonical model for **FOL** + **K** is not a canonical model for **FOL** + **K**.

FOL + **K** and **FOL** + **K** + *BF*

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Lemma The augmentation of the smallest canonical model for **FOL** + **K** + *BF* is a canonical for **FOL** + **K** + *BF*.

Theorem **FOL** + **K** + *BF* is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

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1. **S4M** is complete for the class of all frames that are reflexive, transitive and *final* (every world can see an 'end-point'). However **FOL** $+$ **S4M** is incomplete for Kripke models based on **S4M**-frames. (see Hughes and Cresswell, pg. 283).

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2. **S4.2** is S4 with $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$. This logics is complete for the class of frames that are reflexive, transitive and *convergent*. However, **FOL** $+$ **S4M** $+$ *BF* is incomplete for the class of constant domain models based on reflexive, transitive and convergent frames. (see Hughes and Creswell, pg. 271)

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3. The quantified extension of **GL** is not complete (with respect to varying domains models).

What is going on?

R. Goldblatt. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics*. Lecture Notes in Logic No. 38, Cambridge University Press, 2011.

Background: Incompleteness

There are (consistent) modal logics that are incomplete

A general model is a structure $\langle W, R, V, \mathcal{A}\rangle$ where $\mathcal{A}$ is a suitable boolean algebra with an operator of propositions.

All modal logics are sound and strongly complete with respect to general frames.

Theorem (Goldblatt and Mares) For any **canonical** propositional modal logic **S**, its quantified extension **QS** is complete over a class of general frames for which the underlying propositional frame are just the **S**-frames.

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- New perspective on the Barcan formula: it corresponds to **Tarskian models**
- There is a trade-off between having the underlying Kripke frame validate the propositional logic in question and having a Tarskian-reading of the quantifier.

Central Idea

Algebraic reading of the universal quantifier: $\forall x\varphi$ is true at a world w iff there is some proposition X such that X entails every instantiation of φ and X obtains at w.

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$\mathcal{M}, w \models_s \forall x A$ iff there is a proposition X such that $w \in X$ and $X \subseteq [\![A]\!]_{\mathcal{M},s[x|d]}$ for all $d \in D$.

vs.

$\mathcal{M}, w \models_s \forall x A$ iff for all $d \in D$, $\mathcal{M}, w \models_{s[x|d]} A$

General Frames

Let $\langle W, R \rangle$ be a frame.

$[R] : \wp W \rightarrow \wp W$ where
$[R](X) = \{w \in W \mid$ for all $v \in W$, wRv implies $v \in X\}$
So $[\![\Box\alpha]\!]_{\mathcal{M}} = [R][\![\alpha]\!]_{\mathcal{M}}$

$X \Rightarrow Y = (W - X) \cup Y$
So $[\![\alpha \rightarrow \beta]\!]_{\mathcal{M}} = [\![\alpha]\!]_{\mathcal{M}} \Rightarrow [\![\beta]\!]_{\mathcal{M}}$.

Halmos Functions

$\varphi : D^{\mathcal{V}} \to \wp W$
Let φ and ψ be two such functions, we can lift $[R]$ and $\Rightarrow$ to operations of functions: Eg., if $\varphi : D^{\mathcal{V}} \to \wp W$ and $f \in D^{\mathcal{V}}$.
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Fix a set $Prop \subseteq \wp W$. This defines for each $S \subseteq \wp W$,

$$\sqcap S = \bigcup \{X \in Prop \mid X \subseteq \bigcap S\}$$

General Frames for First-Order Modal Logic

Suppose $Prop \subseteq \wp W$ and let $\varphi : D^{\mathcal{V}} \to Prop$,
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$\langle W, R, V, Prop, PropFun\rangle$ where

- *Prop* contains $\emptyset$ and is closed under $\Rightarrow$ and $[R]$
- Contains the function $\varphi_\emptyset(f) = \emptyset$ for all $f \in D^{\mathcal{V}}$
- PropFun is closed under $\Rightarrow$, $[R]$ and $\forall_x$.
- Assume $(P)^{\mathcal{M}} : D^{\mathcal{V}} \to \wp W$ is an element of *PropFun* for each atomic predicate P.

General Completeness

Theorem For any propositional modal logic **S**, the quantified logic **QS** is complete for the class of (all validating) quantified general frames.

Note that the canonical model construction has as worlds maximally consistent sets that need not be $\forall$-complete.

Key Results

Theorem (Goldblatt and Mares) If **S** is a canonical propositional logic, then **QS** is characterized by the class of all **QS**-frames whose underlying propositional frames validate **S**.

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Logics containing the Barcan formula have **two** characterizing canonical general frames: one that is Tarskian and one that is not.

1. If **S** is canonical, then the second canonical model will have an underlying propositional frame that validates **S** (eg., **S4.2**), but may not be Tarskian.

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Logics containing the Barcan formula have **two** characterizing canonical general frames: one that is Tarskian and one that is not.

1. If **S** is canonical, then the second canonical model will have an underlying propositional frame that validates **S** (eg., **S4.2**), but may not be Tarskian.

2. On the other hand, The Tarskian canonical model may not have an underlying propositional frame that is a frame for **S** (again **S4.2** is an example).