Introduction to Possibility Semantics

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Partial-State Frame

A **partial-state frame** is a tuple $\mathcal{F} = \langle S, R, \sqsubseteq, P\rangle$ where

1. S is a nonempty set (the set of states)
2. $\sqsubseteq$ is a partial order on S (the **refinement** relation)
3. R is a binary relation on S (the accessibility relation — possibly more than one)
4. P is a subset of $\wp(S)$ such that $\emptyset \in P$ and for all $X, Y \in P$:

   4.1 $X \cap Y \in P$
   4.2 $X \supset Y = \{s \in S \mid \forall s' \sqsubseteq s : s' \in X \Rightarrow s' \in Y\} \in\in P$
   4.3 $\blacksquare Y = \{s \in S \mid R(s) \subseteq Y\} \in P$

A model is a tuple $\langle \mathcal{F}, \pi\rangle$ where $\pi : \mathrm{At} \to P$.

$x \sqsubseteq y$ means that the state x is a *refinement* or *further specification* or *extension* of the state y

Suppose that $\mathcal{M} = \langle S, R, \sqsubseteq, P, \pi \rangle$ is a partial-state model with $x \in S$:

- $\mathcal{M}, x \models p$ iff $x \in \pi(p)$
- $\mathcal{M}, x \models \neg\varphi$ iff $\forall x' \sqsubseteq x$, $\mathcal{M}, x' \not\models \varphi$
- $\mathcal{M}, x \models \varphi \wedge \psi$ iff $\mathcal{M}, x \models \varphi$ and $\mathcal{M}, x \models \psi$
- $\mathcal{M}, x \models \varphi \rightarrow \psi$ iff $\forall x' \sqsubseteq x$, if $\mathcal{M}, x' \models \varphi$ then $\mathcal{M}, x' \models \psi$
- $\mathcal{M}, x \models \Box\varphi$ iff $\forall y \in R(x)$, $\mathcal{M}, y \models \varphi$

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- $\mathcal{M}, x \models \varphi \rightarrow \psi$ iff $\forall x' \sqsubseteq x$, if $\mathcal{M}, x' \models \varphi$ then $\mathcal{M}, x' \models \psi$
- $\mathcal{M}, x \models \Box\varphi$ iff $\forall y \in R(x)$, $\mathcal{M}, y \models \varphi$

Fact: Given $\varphi \vee \psi \ := \ \neg(\neg\varphi \wedge \neg\psi)$,

$\mathcal{M}, x \models \varphi \vee \psi$ iff $\forall x' \sqsubseteq x\ \exists x'' \sqsubseteq x'$, $\mathcal{M}, x'' \models \varphi$ or $\mathcal{M}, x'' \models \psi$

Suppose that $\mathcal{F} = \langle S, R, \sqsubseteq, P\rangle$ is a partial-state frame and $\mathcal{M} = \langle \mathcal{F}, \pi\rangle$ a partial-state model:

1. $[\![p]\!]_{\mathcal{M}} = \pi(p)$
2. $[\![\neg\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \supset [\![\emptyset]\!]_{\mathcal{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$
4. $[\![\varphi \rightarrow \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \supset [\![\psi]\!]_{\mathcal{M}}$
5. $[\![\Box\varphi]\!]_{\mathcal{M}} = \blacksquare[\![\varphi]\!]_{\mathcal{M}}$

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- For any formula $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathcal{M}} \in P$
- The set of formulas valid over $\mathcal{F}$ is closed under uniform substitution

World Frames

A relational frame $\langle W, R \rangle$ can be regarded as a partial-frame $\langle W, R, \sqsubseteq, P \rangle$ where

1. $\sqsubseteq$ is the identity relation
2. $P = \wp(W)$

A general relational frame $\langle W, R, A \rangle$ can be regarded as a partial-frame $\langle W, R, \sqsubseteq, A \rangle$ where

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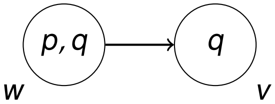
1. $\sqsubseteq$ is the identity relation

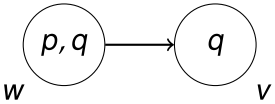
Fact. The definition of truth for Boolean connectives reduces to the standard definition on world frames.

Powerset Possibilization

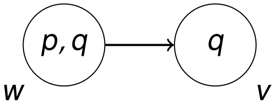
Given a world frame $\mathfrak{F} = \langle W, R, A \rangle$ and a world model $\mathcal{M} = \langle \mathfrak{F}, V \rangle$, the powerset possibilization are $\mathfrak{F}^{\wp} = \langle S, \sqsubseteq, R, P \rangle$ and $\mathfrak{M}^{\wp} = \langle \mathfrak{F}^{\wp}, \pi \rangle$, defined as follows:

1. $S = \wp(W) - \emptyset$
2. $X \sqsubseteq Y$ iff $X \subseteq Y$
3. XRY iff $Y \subseteq R[X]$
4. $P = \{\downarrow X \mid X \in A\}$
5. $\pi(p) = \{X \in S \mid X \subseteq V(p)\}$


p, q
q
w
v

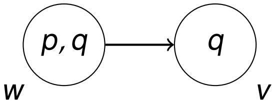


$V(p) = \{w\} \qquad V(q) = \{v\}$



$$V(p) = \{w\} \qquad V(q) = \{v\}$$

$$[\![\Box q]\!]_{\mathcal{M}} = \{w, v\} \qquad [\![\neg q]\!]_{\mathcal{M}} = \emptyset \quad [\![q \rightarrow p]\!]_{\mathcal{M}} = \{w\}$$



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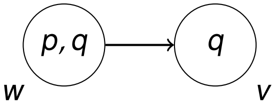
$$[\![\Box q]\!]_{\mathcal{M}} = \{w, v\} \qquad [\![\neg q]\!]_{\mathcal{M}} = \emptyset \quad [\![q \to p]\!]_{\mathcal{M}} = \{w\}$$

$$\begin{tikzcd}
 & \{w,v\} \arrow[dl, "\sqsubseteq"'] \arrow[dr, "\sqsubseteq"] \arrow[dr, dashed, bend left, "R"] & \\
\{w\} \arrow[rr, dashed, "R"'] & & \{v\}
\end{tikzcd}$$



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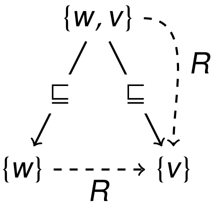
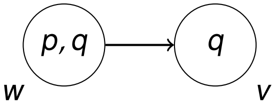


$$\pi(p) = \{\{w\}\} \quad \pi(q) = \{\{w\}, \{v\}, \{w, v\}\}$$



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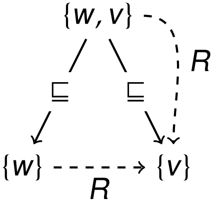

$$\pi(p) = \{\{w\}\} \quad \pi(q) = \{\{w\}, \{v\}, \{w, v\}\}$$
$$[\![\Box q]\!]_{\mathcal{M}^{\wp}} = \{\{w\}, \{v\}, \{w, v\}\}\} \quad [\![\neg q]\!]_{\mathcal{M}^{\wp}} = \{\emptyset\} \quad [\![q \to p]\!]_{\mathcal{M}^{\wp}} = \{\{w\}\}$$

Fact.

1. For any $X \in \mathfrak{M}^{\wp}$ and $\varphi \in \mathcal{L}$, $\mathfrak{M}^{\wp}, X \models \varphi$ iff $\forall x \in \mathfrak{M}$, $\mathfrak{M}, x \models \varphi$
2. For any set of formulas Σ, Σ is satisfiable over $\mathfrak{F}^{\wp}$ iff Σ is satisfiable over $\mathfrak{F}$

Corollary. **K** is sound with respect to the class of all powerset possibilizations of world frames and complete with respect to the class of powerset possibilizations of full world frames.

Fact.

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Corollary. **K** is sound with respect to the class of all powerset possibilizations of world frames and complete with respect to the class of powerset possibilizations of full world frames. Moreover, any normal modal logic that is sound and complete with respect to a class $\mathbb{F}$ of world frames, according to standard Kripke semantics, is also sound and complete with respect to the class of powerset possibilizations of frames from $\mathbb{F}$, according to partial-state semantics.

Possibility frames

Note that $\varphi \leftrightarrow \neg\neg\varphi$ is not valid on partial-state frames.

Refinability: If $\mathcal{M}, x \not\models \varphi$ then there is a $x' \sqsubseteq x$ such that $\mathcal{M}, x' \models \neg\varphi$

If φ is indeterminate at x, i.e., if $\mathcal{M}, x \not\models \varphi$ and $\mathcal{M}, x \not\models \neg\varphi$, then there is a refinement of x that decides φ negatively and there is a refinement of x that decides φ affirmatively.

Indeterminacy of φ is equivalent to having refinements that decide φ each way.

Persistence: if $\mathcal{M}, x \models \varphi$ and $x' \sqsubseteq x$, then $\mathcal{M}, x' \models \varphi$.

$$x \models \neg\neg\varphi$$

$$
\begin{tikzcd}
x \models \neg\neg\varphi \arrow[d, dash, "\forall"] \\
y \not\models \neg\varphi
\end{tikzcd}
$$

$$
\begin{tikzcd}
x \models \neg\neg\varphi \arrow[d, dash, "\forall"] \\
y \not\models \neg\varphi \arrow[d, dash, "\exists"] \\
z \models \varphi
\end{tikzcd}
$$

persistence

$$\begin{tikzcd} x \models \varphi \arrow[d, dash, "\forall"] \\ y \models \varphi \end{tikzcd} \qquad \begin{tikzcd} x \models \neg\neg\varphi \arrow[d, dash, "\forall"] \\ y \not\models \neg\varphi \arrow[d, dash, "\exists"] \\ z \models \varphi \end{tikzcd}$$

Persistence implies that $\varphi \to \neg\neg\varphi$ is valid persistence

$$\begin{array}{ll} x & \models \varphi \\ \Big| & \forall \\ y & \models \varphi \end{array}$$

$$\begin{array}{ll} x & \models \neg\neg\varphi \\ \Big| & \forall \\ y & \not\models \neg\varphi \\ \Big| & \exists \\ z & \models \varphi \end{array}$$

refinability

$$\begin{array}{ll} x & \not\models \varphi \\ \Big| & \exists \\ y & \models \neg\varphi \\ \Big| & \forall \\ z & \not\models \varphi \end{array}$$

Persistence implies that $\varphi \to \neg\neg\varphi$ is valid

Refinability implies that $\neg\neg\varphi \to \varphi$ is valid

In classical partial-state frames, every admissible proposition $X \in P$ will satisfy:

- Persistence: if $x \in X$ and $x' \sqsubseteq x$, then $x' \in X$
- Refinability: if $x \notin X$ then $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x' :\ x'' \notin X$

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In intuitionistic models, the admissible propositions are all the downsets, but in classical models, the admissible propositions are all downsets that also satisfy admissibility.

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X satisfies both persistence and refinability is equivalent to X satisfying:

$x \in X$ iff $\forall x' \sqsubseteq x' \; \exists x'' \sqsubseteq x' : \; x'' \in X$

Let $\mathcal{O}(S, \sqsubseteq)$ be the set of all downsets in $\langle S, \sqsubseteq \rangle$.

$\langle S, \mathcal{O}(S, \sqsubseteq)\rangle$ is a topology (the downset, or Alexandrov, topology).

Interior: $int(X)$ is the largest downset included in X
Closure: $cl(X)$ is the smallest upset that includes X

$$[\![\neg\varphi]\!]_{\mathcal{M}} = int(S - [\![\varphi]\!]_{\mathcal{M}})$$
$$[\![\varphi \rightarrow \psi]\!]_{\mathcal{M}} = int((S - [\![\varphi]\!]_{\mathcal{M}}) \cup [\![\psi]\!]_{\mathcal{M}})$$
$$[\![\varphi \vee \psi]\!]_{\mathcal{M}} = int(cl([\![\varphi]\!]_{\mathcal{M}} \cup [\![\psi]\!]_{\mathcal{M}}))$$

A set X is **regular open** of $X = int(cl(X))$.

Fact. For any topological space $\langle S, \mathcal{O} \rangle$, the structure $\langle RO(S), \wedge, -, \top \rangle$ where

$RO(S)$ is the set of all regular open sets in the topology, $X \wedge Y = X \cap Y$, $-X = int(S - X)$, and $\top = S$

is a complete Boolean algebra with

for all $\mathcal{X} \subseteq R(S)$, $\bigwedge \mathcal{X} = int(\bigcap \mathcal{X})$ and $\bigvee \mathcal{X} = int(cl(\bigcup \mathcal{X}))$.

Lemma. For any poset $\langle S, \sqsubseteq \rangle$ and $X \subseteq S$:

1. $int(cl(X)) = \{x \in S \mid \forall x' \sqsubseteq x\ \exists x'' \sqsubseteq x' : \ x'' \in X\}$
2. $int(cl(\Downarrow X))$ is the smallest regular open set that includes X, where $\Downarrow X = \{y \in S \mid \exists x \in X : \ y \sqsubseteq x\}$
3. X satisfies persistence and refinability iff X is regular open in $\mathcal{O}(S, \sqsubseteq)$

Proposition For any partial-state frame $\mathcal{F} = \langle S, \sqsubseteq, R, P\rangle$ the following are equivalent:

1. the set of $\varphi \in \mathcal{L}$ valid over $\mathcal{F}$ is a classical normal modal logic;
2. for every $\varphi \in \mathcal{L}$, $\neg\neg\varphi$ is equivalent to φ over $\mathcal{F}$; and
3. $P \subseteq RO(\mathcal{F})$

Definition. A **possibility frame** is a partial-state frame $\mathcal{F} = \langle S, \sqsubseteq, R, P\rangle$ in which $P \subseteq RO(\mathcal{F})$. A full possibility frame is a possibility frame in which $P = RO(\mathcal{F})$

An important property of a full possibility frame $\mathcal{F}$ is that $RO(\mathcal{F})$ is closed under $\blacksquare$.

This is not trivial, for there are possibility frames $\mathcal{F}$ that lack the property.

By contrast, it is easy to check that for any $\mathcal{F}$, $RO(\mathcal{F})$ is closed under $\cap$ and $\supset$.

The fact that not every possibility frame is such that $RO(\mathcal{F})$ is closed under $\blacksquare$ means that not every possibility frame can be turned into a full possibility frame simply by replacing its set of admissible propositions P by $RO(\mathcal{F})$.

Intuitionistic Modal Frames

A **full intuitionistic modal frame** is a partial-state frame $\mathcal{F} = \langle S, R, \sqsubseteq, P \rangle$ satisfying:

1. up-R: if $x' \sqsubseteq x$ and $x'Ry'$, then xRy'
2. R-down: if $y' \sqsubseteq y$ and xRy, then xRy'
3. P is the set of all downsets in $\langle S, \sqsubseteq \rangle$

Persistence: If $\mathcal{M}, x \models \varphi$ and $x' \sqsubseteq x$, then $\mathcal{M}, x' \models \varphi$.

up-R

If $x' \sqsubseteq x$ and $x' R y'$, then $x R y'$.

$$\begin{tikzcd} x \arrow[d] & \\ x' & y' \end{tikzcd}$$

up-R

If $x' \sqsubseteq x$ and $x' R y'$, then $x R y'$.

$$\begin{tikzcd} x \arrow[d] & \\ x' \arrow[r, dashed, "R"] & y' \end{tikzcd}$$

up-R

If $x' \sqsubseteq x$ and $x' R y'$, then $x R y'$.

$$\begin{tikzcd}
x \arrow[d] \arrow[dr, dashed, "R"] & \\
x' \arrow[r, dashed, "R"] & y'
\end{tikzcd}$$

$$x \in \blacksquare X$$

$$x' \notin \blacksquare X$$

Claim. If $x' \sqsubseteq x$ and $x \in \blacksquare X$, then $x' \in \blacksquare X$

$$
\begin{tikzcd}
x \in \blacksquare X \arrow[dr, dash] & \\
& x' \notin \blacksquare X \arrow[dl, dashed] \\
y' \notin X &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
x \in \blacksquare X \arrow[dr, dash] \arrow[dd, dashed] & \\
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Up-R: If $x' \sqsubseteq x$ and $x'Ry'$, then xRy'.

For any poset $\langle S, \sqsubseteq\rangle$ and binary relation R on S the following are equivalent:

1. $RO(S, \sqsubseteq)$ is closed under $\blacksquare$
2. R and $\sqsubseteq$ and satisfy:
   - 2.1 R-rule: if $x' \sqsubseteq x$ and $x'\ R\ y' \between z$, then $\exists y :\ x\ R\ y \between z$
   - 2.2 R$\Rightarrow$win: if xRy, then $\forall y' \sqsubseteq y\ \exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x'\ \exists y'' \between y' :\ x''Ry''$

- If $\mathcal{F}$ satisfies R-rule, then $\mathcal{F}$ satisfies up-R
- If $\mathcal{F}$ satisfies down-R, then R$\Rightarrow$win is equivalent with: R-refinability: if xRy then $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x'\ \exists y' \sqsubseteq y :\ x''Ry'$

Claim. If $x' \sqsubseteq x$ and $x \in \blacksquare X$, then $x' \in \blacksquare X$

$$\begin{array}{c} x \in \blacksquare X \\ \Big| \\ x' \notin \blacksquare X \end{array}$$

Claim. If $x' \sqsubseteq x$ and $x \in \blacksquare X$, then $x' \in \blacksquare X$

$$\begin{tikzcd}
x \in \blacksquare X \arrow[d, dash] & \\
x' \notin \blacksquare X \arrow[dr, dash, dashed] & \\
& y' \notin X
\end{tikzcd}$$

Claim. If $x' \sqsubseteq x$ and $x \in \blacksquare X$, then $x' \in \blacksquare X$

refinability

$$\begin{tikzcd}[row sep=large]
x \notin X \arrow[d, dash, "\exists"] \\
y \arrow[d, dash, "\forall"] \\
z \notin X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[row sep=large]
x \in \blacksquare X \arrow[d, dash] & \\
x' \notin \blacksquare X \arrow[dr, dash, dashed] & \\
& y' \notin X \arrow[d, dash, "\exists"] \\
& z \arrow[d, dash, "\forall"] \\
& z' \notin X
\end{tikzcd}$$

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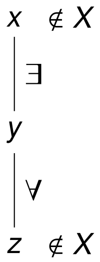
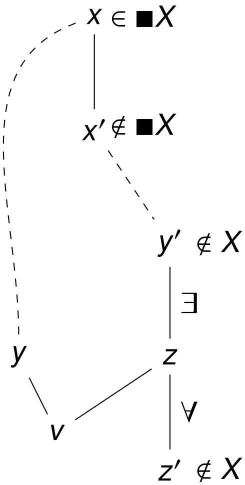
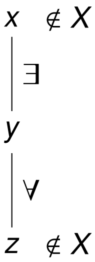


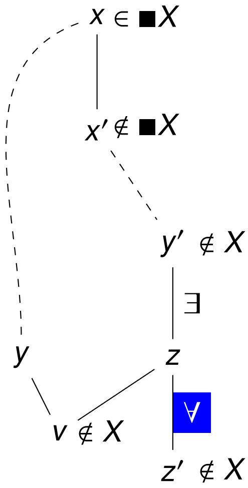


Claim. If $x' \sqsubseteq x$ and $x \in \blacksquare X$, then $x' \in \blacksquare X$

R-rule: if $x' \sqsubseteq x$ and $x' \; R \; y' \between z$, then $\exists y : \; x \; R \; y \between z$

refinability





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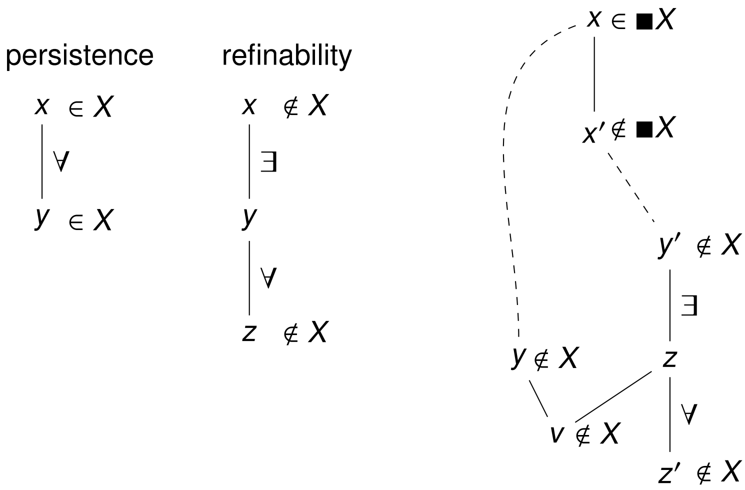

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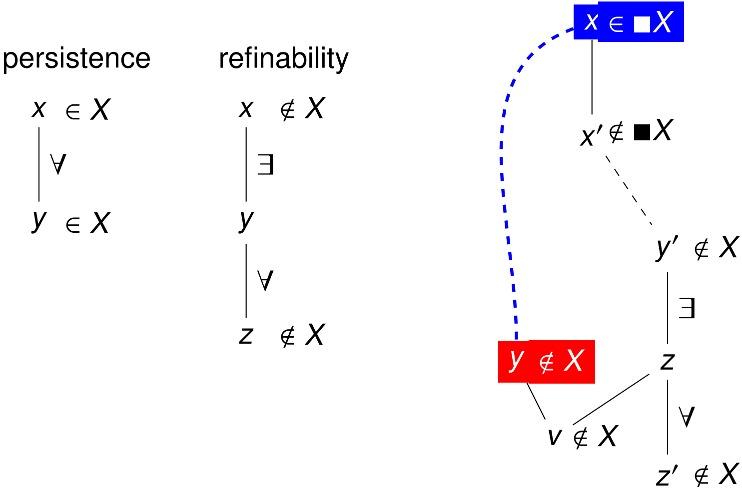

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R-rule: if $x' \sqsubseteq x$ and $x' \; R \; y' \between z$, then $\exists y : \; x \; R \; y \between z$

$x \notin \blacksquare X$

Claim. if $x \notin \blacksquare X$ then $\exists x' \sqsubseteq x \ \forall x'' \sqsubseteq x' : \ x'' \notin \blacksquare X$

$x \notin \blacksquare X$

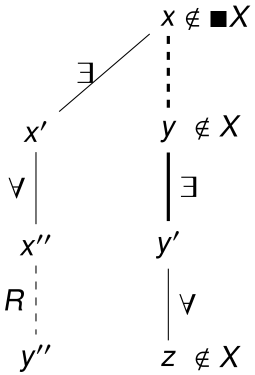
$y \notin X$

Claim. if $x \notin \blacksquare X$ then $\exists x' \sqsubseteq x \; \forall x'' \sqsubseteq x' : \; x'' \notin \blacksquare X$

$$\begin{tikzcd}
x \notin \blacksquare X \arrow[d, dashed, no head] \\
y \notin X \arrow[d, no head, "\exists"] \\
y' \arrow[d, no head, "\forall"] \\
z \notin X
\end{tikzcd}$$

Claim. if $x \notin \blacksquare X$ then $\exists x' \sqsubseteq x \; \forall x'' \sqsubseteq x' : \; x'' \notin \blacksquare X$

Refinability for X



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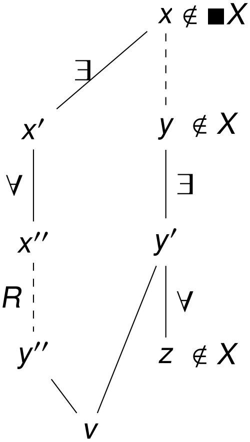
R$\Rightarrow$win: if xRy, then
$\forall y' \sqsubseteq y \; \exists x' \sqsubseteq x \; \forall x'' \sqsubseteq x' \; \exists y'' \between y' : \; x'' R y''$



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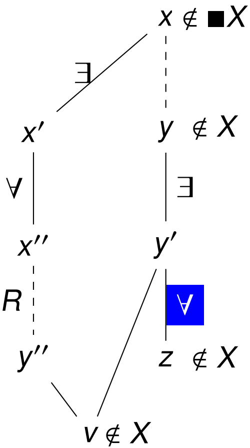

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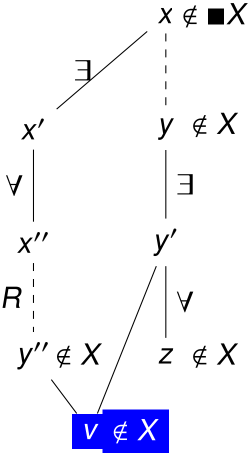

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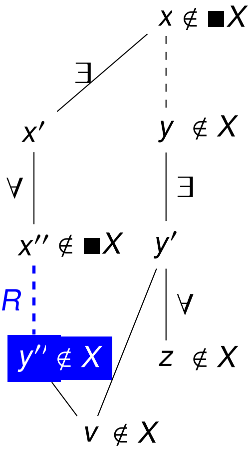

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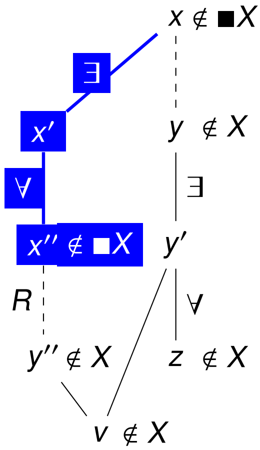

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The are full possibility frames $\mathcal{F}$ that validate a modal formula that is not valid on any Kripke frame. Thus, the logic of $\mathcal{F}$ will be a normal modal logic that is Kripke-frame inconsistent—it is not sound with respect to any Kripke frame.

Suppose that φ and ψ for formulas such that the propositional variable p does not occur in ψ. The consider the following formula:

$$\textsf{SPLIT} \qquad \Diamond_i(p \wedge \psi) \rightarrow (\Diamond_i(p \wedge \varphi) \wedge \Diamond_i(p \wedge \neg\varphi))$$

Any Kripke frame $\mathfrak{F}$ that validates SPLIT must also validate $\neg\Diamond_i\psi$.

Worlds cannot split, but possibilities can: There is a full possibility frames that validates and instance of SPLIT and $\Diamond_i\psi$.

There are three approaches to valuation functions in the literature on possibility semantics.

There are three approaches to valuation functions in the literature on possibility semantics.

The approach followed here: a valuation is a total function $\pi : \mathsf{At} \to \wp(S)$ such that $\pi(p)$ satisfies persistence and refinability.

$x \in \pi(p)$ means that x determines that p is true and $x \notin \pi(p)$ means that x does not determine that p is true, i.e., that either x determines that p is false or x does not determine the truth or falsity of p.

Partial Valuations

A valuation is a partial function $V : \mathrm{At} \times S \to \{0, 1\}$ satisfying stability and resolution:

1. stability $V(p, x)$ is defined and $x' \sqsubseteq x$, then $V(p, x')$ is defined and $V(p, x) = V(p, x')$
2. resolution: if $V(p, x)$ is undefined, then there are $y \sqsubseteq x$ and $z \sqsubseteq x$ such that $V(p, y) = 1$ and $V(p, z) = 0$.

$V(p, x) = 1$ means that x determines that p is true; $V(p, x) = 0$ means that x determines that p is false; $V(p, x)$ being undefined means that x does not determine the truth or falsity of p.

Total Valuations

$U : \mathrm{At} \times S \to 0, 1$ is a total function such that $\{x \in S \mid U(p, x) = 1\}$ satisfies persistence and refinability in the sense of this paper; $U(p, x) = 1$ means that x determines that p is true; $U(p, x) = 0$ means that x does not determine that p is true.

M. Harrison-Trainor. *Worldization of Possibility Models*. manuscript, 2018.

From possibilities to worlds

"[T]he business of making a possibility more determinate seems openended. There are possibilities that the child at home should be a boy, a six-year-old boy, a six-year-old boy with blue eyes, a six-year old boy with blue eyes who weighs 3 stone, and so forth. So far from terminating in a fully determinate possibility, we seem to have an indefinitely long sequence of increasingly determinate possibilities, any one of which is open to further determination. But then, so far from conceiving of our rational activities as discriminating between regions of determinate points, we appear to have no clear conception of such a point at all. "

I. Rumfitt. *The Boundary Stones of Thought: An Essay in the Philosophy of Logic*. Oxford University Press, 2015.

Theorem (Harrison-Trainor). Let $\mathcal{M}$ be a countable possibility model in a countable language. Then there is a Kripke model $\mathcal{K}$ which is a **worldization** of $\mathcal{M}$.

Theorem (Harrison-Trainor). Let $\mathcal{M}$ be a countable possibility model in a countable language. Then there is a Kripke model $\mathcal{K}$ which is a **worldization** of $\mathcal{M}$.

Note: there are counterexamples if $\mathcal{M}$ is not countable or the language is not countable.

J. van Benthem, N. Bezhanishvili and W. Holliday. *A Bimodal Perspective on Possibility Semantics*. Journal of Logic and Computation, 27(5), 2017, pp. 1353 - 1389.