Modal Logic

PHIL 858P

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Let $\mathsf{At} = \{p_1, p_2, \ldots, p_n, \ldots\}$ and $\mathcal{L}$ be the basic modal language:

$$p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi$$

where $p \in \mathsf{At}$ is a propositional variable.

The **height** of $\varphi \in \mathcal{L}_\Diamond$, denoted $\mathsf{ht}(\varphi)$, is:

$$\begin{array}{lcl} \mathsf{ht}(p_n) & = & 0 \\ \mathsf{ht}(\neg\varphi) & = & \mathsf{ht}(\varphi) \\ \mathsf{ht}(\varphi \vee \psi) & = & \max\{\mathsf{ht}(\varphi), \mathsf{ht}(\psi)\} \\ \mathsf{ht}(\Diamond\varphi) & = & 1 + \mathsf{ht}(\varphi) \end{array}$$

The **order** of φ, written $\mathsf{ord}(\varphi)$, is

$$\begin{array}{lcl} \mathsf{ord}(p_n) & = & n \\ \mathsf{ord}(\neg\varphi) & = & \mathsf{ord}(\varphi) \\ \mathsf{ord}(\varphi \vee \psi) & = & \max\{\mathsf{ord}(\varphi), \mathsf{ord}(\psi)\} \\ \mathsf{ord}(\Diamond_n\varphi) & = & \mathsf{ord}(\varphi) \end{array}$$

$$\mathcal{L}_{h,n} = \{\varphi \mid \varphi \in \mathcal{L},\ \mathsf{ht}(\varphi) \leq h \text{ and } \mathsf{ord}(\varphi) \leq n\}$$

Propositional Logic

$\mathcal{L}_{0,n}$ is the propositional language built from $\{p_1, \ldots, p_n\}$ of propositional variables.

For any $T \subseteq \{p_1, \ldots, p_m\}$, let

$$\widehat{T} = \bigwedge_{p \in T} p \wedge \bigwedge_{p \in \{p_1, \ldots, p_n\} - T} \neg p$$

- For each $\varphi \in \mathcal{L}_{0,m}$, exactly one of the following holds: $\vdash \widehat{T} \rightarrow \varphi$ or $\vdash \widehat{T} \rightarrow \neg\varphi$.
- For each $\varphi \in \mathcal{L}_{0,m}$, $\vdash \varphi \leftrightarrow \bigvee\{\widehat{T} \mid \ \vdash \widehat{T} \rightarrow \varphi\}$.

Canonical sentences

$$\begin{array}{rcl} \mathcal{C}_{0,n} & = & \{\widehat{T} \mid T \subseteq \{p_1, \ldots, p_n\}\} \\ \mathcal{C}_{h+1,n} & = & \{\alpha_{S,T} \mid S \subseteq \mathcal{C}_{h,n},\ T \subseteq \{p_1, \ldots, p_n\}\} \end{array}$$

where

$$\alpha_{S,T} := \bigwedge_{\psi \in S} \Diamond\psi \ \wedge\ \Box \bigvee S \ \wedge\ \widehat{T}$$

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$$\begin{array}{lclcl} \alpha_1 & = & \widehat{\emptyset} \wedge p & = & \Box\bot \wedge p \\ \alpha_2 & = & \widehat{\emptyset} \wedge \neg p & = & \Box\bot \wedge \neg p \end{array}$$

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\alpha_7 & = & \widehat{\mathcal{C}_{0,1}} \wedge p & = & \Diamond p \wedge \Diamond\neg p \wedge \Box(p \vee \neg p) \wedge p \\
\alpha_8 & = & \widehat{\mathcal{C}_{0,1}} \wedge \neg p & = & \Diamond p \wedge \Diamond\neg p \wedge \Box(p \vee \neg p) \wedge \neg p
\end{array}
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Lemma. For each h and n, $\mathcal{C}_{h,n}$ is a finite subset of $L_{h,n}$. Moreover, if $F(0,n) = 2^n$ and $F(h+1,n) = 2^{F(h,n)+n}$, then $|\mathcal{C}_{h,n}| = F(h,n)$

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Lemma. Let $\chi \in \mathcal{L}_{h,n}$ and $\alpha \in \mathcal{C}_{h,n}$. Then, either $\vdash_{\mathbf{K}} \alpha \rightarrow \chi$ or $\vdash_{\mathbf{K}} \alpha \rightarrow \neg\chi$.

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Definition. Given a set of formulas X, let $\bigoplus X$ denote *exactly one of* X. Formally, if $X = \{\varphi_1, \ldots, \varphi_n\}$, then $\bigoplus X$ is short for $\bigvee_{i=1,\ldots,n}(\varphi_i \wedge \neg \bigvee_{j\neq i} \varphi_j)$.

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$\vdash_{\mathbf{K}} \varphi \leftrightarrow \bigvee\{\alpha \mid \alpha \in \mathcal{C}_{h,n},\ \vdash_{\mathbf{K}} \alpha \to \varphi\}$

Canonical Model

Canonical Model: $\mathcal{M}^C = \langle W^C, R^C, V^C \rangle$ for **L**, where

$W^C = \{\Gamma \mid \Gamma$ is a maximally **L**-consistent set$\}$

$\Gamma R^C \Delta$ iff $\{\alpha \mid \Box\alpha \in \Gamma\} \subseteq \Delta$

$V^C(p) = \{\Gamma \mid p \in \Gamma\}$

Fact: $\Gamma R^C \Delta$ iff for all $\varphi \in \mathcal{L}$, if $\varphi \in \Delta$, then $\Diamond\varphi \in \Gamma$.

Truth Lemma: For all Γ, for all $\varphi \in \mathcal{L}$, $\mathcal{M}^C, \Gamma \models \varphi$ iff $\varphi \in \Gamma$.

Finite Canonical Model

$\mathbb{C}_{h,n}(\mathbf{L}) = \langle W^c, R^c, V^c \rangle$ for **L**, where

$W^c = \mathcal{C}_{h,n}(\mathbf{L}) = \{\alpha \mid \alpha \in \mathcal{C}_{h,n}$ and α is $\mathbf{L}$-consistent$\}$

$\alpha R^c \beta$ iff $\alpha \wedge \Diamond\beta$ is **L**-consistent

$V^c(p) = \{\alpha \mid \vdash_{\mathbf{L}} \alpha \rightarrow p\}$

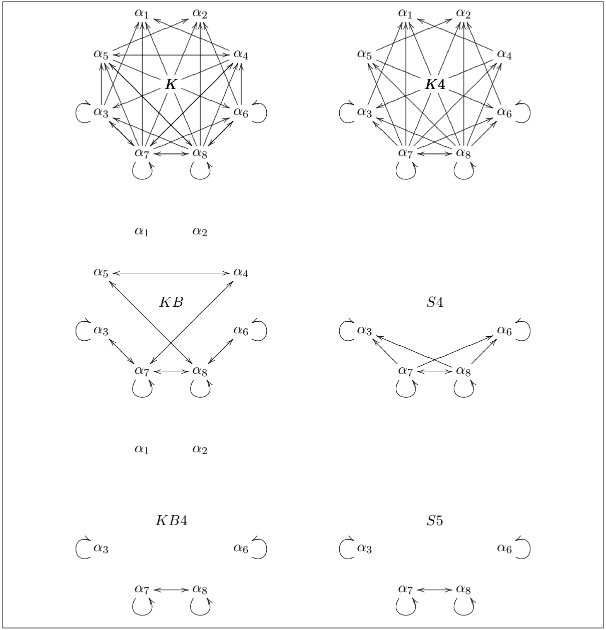

Figure 1: $\mathbb{C}_{1,1}(L)$ for various logics L. The formulas α_1, …, α_8 are from Example 2.1.

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Truth Lemma. For all $\alpha \in \mathcal{C}_{h,n}(\mathbf{L})$ and all $\psi \in \mathcal{L}_{h,n}$, $\mathbb{C}_{h,n}(\mathbf{L}), \alpha \models \psi$ iff $\vdash_{\mathbf{K}} \alpha \to \psi$.

Existence Lemma. Let $\psi \in \mathcal{L}_{h,n}$ and φ be an aribtrary formula and suppose that $\varphi \wedge \Diamond\psi$ is consistent in **L**. Then there is some $\alpha \in \mathcal{C}_{h,h}(\mathbf{L})$ such that $\varphi \wedge \Diamond\alpha$ is consistent in **L** and $\vdash_{\mathbf{K}} \alpha \to \psi$

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1. If $\mathbf{KT} \leq \mathbf{L}$, then $\mathbb{C}_{h,n}$ is reflexive.
2. If $\mathbf{KD} \leq \mathbf{L}$, then $\mathbb{C}_{h,n}$ is serial.
3. If $\mathbf{KB} \leq \mathbf{L}$, then $\mathbb{C}_{h,n}$ is symmetric.

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Weak Completeness and Decidability. If ψ holds at every world in every symmetric model, then $\vdash_{\mathbf{KB}} \psi$. Moreover, the property of being provable in **KB** is decidable.

Transitivity

The completeness results for logics containing **K4** is more difficult.

We must show that if $\alpha \wedge \Diamond\beta$ and $\beta \wedge \Diamond\gamma$ are both consistent in **K4**, then so is $\alpha \wedge \Diamond\gamma$.

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This is not true in general: $(p \wedge \Box\Diamond p) \wedge \Diamond\neg p$ and $\neg p \wedge \Diamond\Box\neg p$ are consistent is **K4**, but $(p \wedge \Box\Diamond p) \wedge \Diamond\Box\neg p$ is not consistent in **K4**.

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But it is true for $\alpha, \beta, \gamma \in \mathbb{C}_{h,n}(\mathbf{K4})$

Open question: Is it true that for all **L**, if $\mathbf{K4} \leq \mathbf{L}$, then $\mathbb{C}_{h,n}(\mathbf{L})$ is transitive?

Lemma. For every $\alpha \in \mathcal{C}_{h+1,n}$, there is a unique $\alpha' \in \mathcal{C}_{h,n}$, called the **derivative** of α, such that $\vdash_{\mathbf{K}} \alpha \to \alpha'$. Moreover for all logics **L**, if $\alpha \in \mathcal{C}_{h+1,n}(\mathbf{L})$, then $\alpha' \in \mathcal{C}_{h,n}(\mathbf{L})$.

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Lemma. Suppose that $\alpha R^c \beta$ in $\mathbb{C}_{h,n}(\mathbf{L})$, then $\vdash_{\mathbf{K}} \alpha \to \Diamond\beta'$.

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Lemma. Let $\mathbf{K4} \leq \mathbf{L}$ and let $\alpha R^c \beta$ in $\mathbb{C}_{h+1,n}(\mathbf{L})$. Then, if $\varphi \in \mathcal{L}_{h,n}$ and $\vdash_{\mathbf{K4}} \beta \to \Diamond \varphi$, then $\vdash_{\mathbf{K4}} \alpha \to \Diamond \varphi$

1. $\mathbb{M}_{0,n}(\mathbf{L}) = \mathbb{C}_{0,n}(\mathbf{L})$
2. The valuation on $\mathbb{M}_{h+1,n}(\mathbf{L})$ is $V(p_i) = \{\alpha \mid \vdash \alpha \rightarrow p_i\}$
3. The accessibility relation R^m in $\mathbb{M}_{h+1,n}(\mathbf{L})$ is defined by $\alpha R^m \beta$ iff
    - 3.1 $\vdash \alpha \rightarrow \Diamond \beta'$
    - 3.2 for all $\gamma \in \mathcal{C}_{h,n}$, if $\vdash \beta \rightarrow \Diamond\gamma$, then $\vdash \alpha \rightarrow \Diamond\gamma$

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Lemma. If $\mathbf{K4} \leq \mathbf{L}$, then if $\alpha R^c \beta$, then $\alpha R^m \beta$

A relation $\rightsquigarrow$ on $\mathcal{C}_{h,n}(\mathbf{L})$ is **suitable** if the following two properties hold:

1. If $\alpha R^c \beta$ in $\mathbb{C}_{h,n}(\mathbf{L})$, then $\alpha \rightsquigarrow \beta$
2. If $\alpha \rightsquigarrow \beta$, then $\vdash \alpha \to \beta'$

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Generalized Truth Lemma Let $\rightsquigarrow$ be a suitable relation on $\mathcal{C}_{h,n}(\mathbf{L})$. Then, for all $\alpha \in \mathcal{C}_{h,n}(\mathbf{L})$ and all $\psi \in \mathcal{L}_{h,n}$,

$$((\mathcal{C}_{h,n}, \rightsquigarrow), \alpha \models \psi \quad \text{iff} \quad \vdash_{\mathbf{K}} \alpha \rightarrow \psi$$

Lemma. $\mathbb{C}_{h,n}(\mathbf{K4}) = \mathbb{M}_{h,n}(\mathbf{L})$.

Theorem. Every $\alpha \in \mathcal{C}_{h,n}(\mathbf{K4})$ is satisfiable in some finite transitive model. Thus, **K4** is complete for transitive models, and also decidable.

KL

$\mathbb{D}_{h,n}(\mathbf{L}) = \langle \mathcal{C}_{h,n}(\mathbf{L}), R^d \rangle$ where

1. $\alpha R^d \beta$ if $\alpha R^c \beta$, and for some $\varphi \in \mathcal{L}_{h,n}$, $\vdash \alpha \rightarrow \Diamond\varphi$ and $\vdash \beta \rightarrow \Box\neg\varphi$.

$\mathbb{N}^*_{h,n}(\mathbf{L}) = \langle \mathcal{C}_{h,n}(\mathbf{L}), R^{n^*} \rangle$ where

1. $\mathbb{N}^*_{0,n}(\mathbf{L}) = \mathbb{C}_{0,n}(\mathbf{L})$
2. The valuation on $\mathbb{N}^*_{h+1,n}(\mathbf{L})$ is $V(p_i) = \{\alpha \mid\ \vdash \alpha \rightarrow p_i\}$
3. The accessibility relation R^m in $\mathbb{N}^*_{h+1,n}(\mathbf{L})$ is defined by $\alpha R^{n^*} \beta$ iff
   3.1 $\vdash \alpha \rightarrow \Diamond\beta'$
   3.2 for all $\gamma \in \mathcal{C}_{h,n}$, if $\vdash \beta \rightarrow \Diamond\gamma$, then $\vdash \alpha \rightarrow \Diamond\gamma$
   3.3 there is some $\gamma \in \mathcal{C}_{h,n}$ such that $\vdash \alpha \rightarrow \Diamond\gamma$, but $\vdash \beta \rightarrow \Box\neg\gamma$

Truth Lemma. Let **KL** $\leq$ **L**. For all $\alpha \in \mathcal{C}_{h,n}(\mathbf{L})$ and all $\psi \in \mathcal{L}_{h,n}$,

$$\mathbb{N}^*_{h,n}(\mathbf{L}), \alpha \models \psi \quad \text{iff} \quad \vdash_{\mathbf{K}} \alpha \to \psi$$

Theorem.

1. $\mathbb{N}^*_{h,n}(\mathbf{KL}) = \mathbb{D}_{h,n}(\mathbf{KL})$.
2. **KL** is complete for finite transitive, converse wellfounded models.
3. $\mathbb{C}_{h,n}(\mathbf{KL})$ is transitive.

Filtrations

Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. Suppose that Σ is a set of formulas closed under subformulas. We write say w and v are Σ-equivalent provided:

$$w \leftrightsquigarrow_\Sigma v \text{ iff for all } \varphi \in \Sigma,\ \mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}, v \models \varphi.$$

Note that $\leftrightsquigarrow_\Sigma$ is an equivalence relation. Let $|w|_\Sigma = \{v \mid w \leftrightsquigarrow_\Sigma v\}$ denote the equivalence class of w under $\leftrightsquigarrow_\Sigma$.

Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. Given a set of formulas Σ closed under subformulas, a model $\mathcal{M}^f = \langle W^f, R^f, V^f \rangle$ is a filtration of $\mathcal{M}$ through Σ provided

- $W^f = \{|w|_\Sigma \mid w \in W\}$
- If wRv then $|w|_\Sigma R^f |v|_\Sigma$
- If $|w|_\Sigma R^f |v|_\Sigma$ then for each $\Diamond\varphi \in \Sigma$, if $\mathcal{M}, v \models \varphi$ then $\mathcal{M}, w \models \Diamond\varphi$
- $V^f(p) = \{|w|_\Sigma \mid w \in V(p)\}$ $\triangleleft$

Theorem. If $\mathcal{M}^f$ is a filtration of $\mathcal{M}$ through Σ, then for all $\varphi \in \Sigma$,

$$\mathcal{M}, w \models \varphi \quad \text{iff} \quad \mathcal{M}^f, |w|_\Sigma \models \varphi$$

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Examples of Filtrations

- **smallest filtration**: $|w|_\Sigma R^s |v|_\Sigma$ iff there is $w' \in |w|_\Sigma$ and $v' \in |v|_\Sigma$ such that $w' R v'$.
- **largest filtration**: $|w|_\Sigma R^l |v|_\Sigma$ iff for all $\Diamond\varphi \in \Sigma$, $\mathcal{M}, v \models \varphi$ implies $\mathcal{M}, w \models \Diamond\varphi$
- **transitive filtration**: $|w|_\Sigma R^t |v|_\Sigma$ iff for all $\Diamond\varphi \in \Sigma$, $\mathcal{M}, v \models \varphi \vee \Diamond\varphi$ implies $\mathcal{M}, w \models \Diamond\varphi$ (assuming R is transitive)

Canonical Model: $\mathcal{M}^C = \langle W^C, R^C, V^C \rangle$ for **L**, where

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We also have $\Gamma \equiv \Delta$ iff $\Gamma \cap \mathcal{L}_{h,n} = \Delta \cap \mathcal{L}_{h,n}$.

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$\Gamma R^C \Delta$ iff $\{\alpha \mid \Box\alpha \in \Gamma\} \subseteq \Delta$

$V^C(p) = \{\Gamma \mid p \in \Gamma\}$

For all $\Gamma \in W^C$, there is a unique $\alpha \in \mathcal{C}_{h,n}$ such that $\alpha \in \Gamma$ (write α_Γ for this formula)

The set $\mathcal{L}_{h,n}$ induces an equivalence relation $\equiv$ on W^C, where $\Gamma \equiv \Delta$ iff $\alpha_\Gamma = \alpha_\Delta$.

We also have $\Gamma \equiv \Delta$ iff $\Gamma \cap \mathcal{L}_{h,n} = \Delta \cap \mathcal{L}_{h,n}$.

For each $\alpha \in \mathcal{C}_{h,n}(\mathbf{L})$, W^C contains $Th_{\mathbb{C}}(\alpha)$, the set of formulas satisfied by α in $\mathbb{C}_{h,n}$.

Theorem. There is a one-to-one correspondence between filtrations of $\mathcal{M}^C$ for $\mathbf{L}$ through $\mathcal{L}_{h,n}$ and suitable relation $\rightsquigarrow$ on $\mathcal{C}_{h,n}(\mathbf{L})$. The correspondence associates to a filtration R^f the suitable relation $\rightsquigarrow_{R^f}$ give by:

$$\alpha \rightsquigarrow_{R^f} \beta \quad \text{iff} \quad [Th(\alpha)]R^f[Th(\beta)]$$

In the other direction, we associate to a suitable relation $\rightsquigarrow$ the filtration $R_{\rightsquigarrow}$ given by

$$[\Gamma]R_{\rightsquigarrow}[\Delta] \quad \text{iff} \quad \alpha_\Gamma \rightsquigarrow \alpha_\Delta$$

Each of these is monotone with respect to inclusion of relations.

1. The minimal filtration of $\mathcal{L}_{h,n}$ corresponds to the accessibility relation R^C of $\mathbb{C}_{h,n}(\mathbf{L})$
2. The largest filtration on $\mathcal{L}_{h,n}$ corresponds to the suitable relation $\rightsquigarrow$ given by $\alpha \rightsquigarrow \beta$ iff $\vdash_{\mathbf{L}} \alpha \rightarrow \beta'$