Lecture 4: Completeness

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1 Frame Definability

Definition 1 (Frame) A pair $\langle W, R\rangle$ with $W \neq \emptyset$ and $R \subseteq W \times W$ is called a **frame**. Given a frame $\mathcal{F} = \langle W, R\rangle$, a model $\mathcal{M}$ **is based on the frame** $\mathcal{F} = \langle W, R\rangle$ if $\mathcal{M} = \langle W, R, V\rangle$ for some valuation function $V : \mathsf{At} \to \mathcal{P}(W)$. $\triangleleft$

Definition 2 (Frame Validity) Given a frame $\mathcal{F} = \langle W, R\rangle$, a modal formula φ is **valid on** $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, provided $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ based on $\mathcal{F}$. $\triangleleft$

Definition 3 (Defining a Class of Frames) A modal formula φ **defines the class of frames with property** P provided for all frames $\mathcal{F}$, $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ has property P. In such a case, we say that φ **corresponds** to P. $\triangleleft$

- Some modal formulas correspond to first-order formulas:
  e.g., $\Box\varphi \to \Box\Box\varphi$ corresponds to transitivity; $\Box\varphi \to \varphi$ corresponds to reflexivity; $\varphi \to \Box\Diamond\varphi$ corresponds to symmetry
- Some modal formulas do not correspond to any first-order formula:
  e.g., $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ and $\Box\Diamond\varphi \to \Diamond\Box\varphi$ do not correspond to any first-order formula.

Definition 4 (p-morphism) A **p-morphism** from $\mathcal{F} = \langle W, R\rangle$ to $\mathcal{F}' = \langle W', R'\rangle$ is a function $f : W \to W'$ such that:

- (**forth**) For all $w, v \in W$, wRv implies that $f(w)R'f(v)$
- (**back**) For all $w \in W$, $w' \in W'$, if $f(w)R'w'$, then there is a $v \in W$ such that wRv and $f(v) = w'$.

We say that $\mathcal{F}'$ is a **p-morphic image** of $\mathcal{F}$ if there is a p-morphism from $\mathcal{F}$ *onto* $\mathcal{F}'$ (so the p-morphism is surjective) $\triangleleft$

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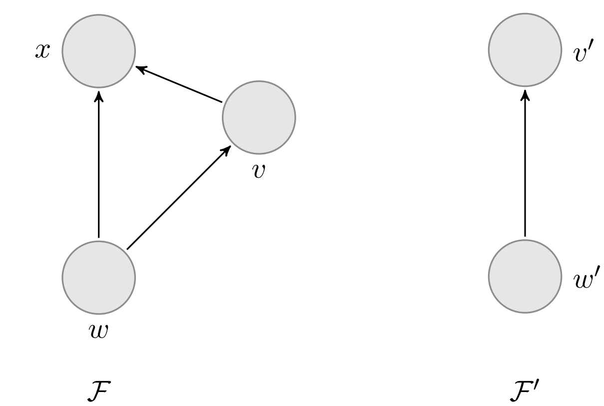
2 Tutorial Questions

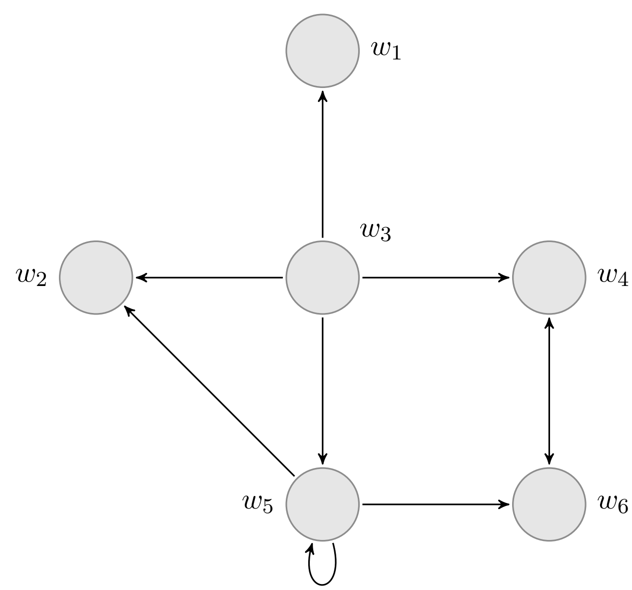
1. Suppose that $\mathcal{F} = \langle W, R \rangle$ and $\mathcal{F}' = \langle W', R' \rangle$ are frames. Prove that $f : W \to W'$ is a p-morphism iff for all $w \in W$,

$$\{f(v) \mid v \in W, wRv\} = \{v' \mid v' \in W', f(w)R'v'\}$$

2. Are there any p-morphisms between these two frames?



3. Find a frame with at most 3 worlds that is a p-morphic image of:



4. Prove that any p-morphic image of a symmetric frame is also symmetric. (Check that the same holds for reflexivity and transitivity.)

5. Suppose that $\mathcal{F} = \langle W, R \rangle$ and $\mathcal{F}' = \langle W', R' \rangle$ are frames and that $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$. Prove that any modal formula that is valid on $\mathcal{F}$ is valid on $\mathcal{F}'$.

6. Prove that irreflexivity $(\forall x, \neg xRx)$ does not correspond to any modal formula. Hint: think of $\langle \mathbb{N}, < \rangle$ and $\langle \{w\}, \{(w, w)\} \rangle$.

7. Suppose that $\mathcal{F} = \langle W, R \rangle$ and $\mathcal{F}' = \langle W', R' \rangle$ are frames. The product $\mathcal{F} \times \mathcal{F}'$ is the frame $\langle W \times W, R^{\times} \rangle$ where $(w, w')R^{\times}(v, v')$ iff wRv and $w'R'v'$. Prove that if at least one of $\mathcal{F}$ and $\mathcal{F}'$ is irreflexive, then $\mathcal{F} \times \mathcal{F}'$ is irreflexive.

8. Let $\mathcal{F} = \langle W, R \rangle$ be any frame and $\mathcal{F}' = \langle W', R' \rangle$ be a serial frame $(\forall x \exists y x R' y)$. Prove that the projection map $\pi : \mathcal{F} \times \mathcal{F}' \to \mathcal{F}$ given by $\pi(w, w') = w$ is a surjective p-morphism.

9. Prove that for all formulas φ, φ is valid on the class of irreflexive frames iff φ is valid on the class of all frames. (So, the modal logic of irreflexive frames is the logic of all frames).

   Hint: First, if φ is valid on all frames, then it must be valid on all irreflexive frames. We must show that if φ is valid on all irreflexive frames, then it is valid on all frames. Suppose φ is valid on all irreflexive frames and let $\mathcal{F}$ be a frame. Use the frame $\mathcal{F} \times \langle \mathbb{N}, < \rangle$ to conclude that φ is valid on $\mathcal{F}$.

3 Modal Axioms

Validity: Suppose that $\mathcal{F} = \langle W, R \rangle$ is a frame and $\mathcal{M} = \langle W, R, V \rangle$ is a model.

- φ is satisfiable when there is a model $\mathcal{M} = \langle W, R, V \rangle$ with a state $w \in W$ such that $\mathcal{M}, w \models \varphi$
- Valid on a model, $\mathcal{M} \models \varphi$: for all $w \in W$, $\mathcal{M}, w \models \varphi$
- Valid on a frame, $\mathcal{F} \models \varphi$: for all $\mathcal{M}$ based on $\mathcal{F}$, for all $w \in W$, $\mathcal{M}, w \models \varphi$
- Valid at a state on a frame at a state $w \in W$, $\mathcal{F}, w \models \varphi$: for all $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M}, w \models \varphi$
- Valid in a class F of frames, $\models_{\mathsf{F}} \varphi$: for all $\mathcal{F} \in \mathsf{F}$, $\mathcal{F} \models \varphi$

Logical Consequence: Suppose that Γ is a set of modal formulas and F is a class of frames. $\Gamma \models_{\mathsf{F}} \varphi$ iff for all frames $\mathcal{F} \in \mathsf{F}$, for all models based on $\mathcal{M}$, for all w in the domain of $\mathcal{M}$, if $\mathcal{M}, w \models \Gamma$, then $\mathcal{M}, w \models \varphi$.

Modal Deduction with Assumptions: Let Γ be a set of modal formulas. A **modal deduction of** φ **from** Γ, denoted $\Gamma \vdash_{\mathbf{K}} \varphi$ is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. α_i is a tautology
2. $\alpha_i \in \Gamma$
3. α_i is a substitution instance of $\Box(p \to q) \to (\Box p \to \Box q)$
4. α_i is of the form $\Box \alpha_j$ for some $j < i$ and $\vdash_{\mathbf{K}} \alpha_j$
5. α_i follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$).

Soundness/Completeness: Suppose that F is a class of relational frames.

- A logic $\mathbf{L}$ is **sound** with respect to F provided, for all sets of formulas Γ, if $\Gamma \vdash_{\mathbf{L}} \varphi$, then $\Gamma \models_{\mathsf{F}} \varphi$.
- A logic $\mathbf{L}$ is **strongly complete** with respect to F provided for all sets of formulas Γ, if $\Gamma \models_{\mathsf{F}} \varphi$, then $\Gamma \vdash_{\mathbf{L}} \varphi$.
- A logic $\mathbf{L}$ is **weakly complete** with respect to F provided that for all $\varphi \in \mathcal{L}$, if $\models_{\mathsf{F}} \varphi$, then $\vdash_{\mathbf{L}} \varphi$.

Some Axioms

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
D	$\Box\varphi \to \Diamond\varphi$
T	$\Box\varphi \to \varphi$
4	$\Box\varphi \to \Box\Box\varphi$
5	$\neg\Box\varphi \to \Box\neg\Box\varphi$
L	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$

Some Modal Logics

K	$K + PC + Nec$
T	$K + T + PC + Nec$
S4	$K + T + 4 + PC + Nec$
S5	$K + T + 4 + 5 + PC + Nec$
KD45	$K + D + 4 + 5 + PC + Nec$
GL	$K + L + PC + Nec$

Completeness Theorems

- **T** is sound and strongly complete with respect to the class reflexive Kripke frames.
- **S4** is sound and strongly complete with respect to the class reflexive Kripke frames.
- **S5** is sound and strongly complete with respect to the class reflexive Kripke frames.
- **KD45** is sound and strongly complete with respect to the class reflexive Kripke frames.

4 Canonical Model

Notation:

- Let **K** denote the minimal modal logic and $\vdash \varphi$ mean φ is derivable in **K**. If Γ is a set of formulas, we write $\Gamma \vdash \varphi$ if $\vdash (\psi_1 \wedge \cdots \wedge \psi_k) \to \varphi$ for some finite set $\psi_1, \ldots, \psi_k \in \Gamma$.
- Let Γ be a set of formulas. If $\mathcal{F}$ is a frame, then we write $\mathcal{F} \models \Gamma$ for $\mathcal{F} \models \varphi$ for each $\varphi \in \Gamma$. We write $\Gamma \models \varphi$ provided for all frames $\mathcal{F}$, if $\mathcal{F} \models \Gamma$ then $\mathcal{F} \models \varphi$.
- A set of formulas Γ is **consistent** provided $\Gamma \not\vdash \bot$.
- Γ is a **maximally consistent set** if Γ is consistent and for each $\varphi \in \mathcal{L}$ either $\varphi \in \Gamma$ of $\neg\varphi \in \Gamma$. Alternatively, Γ is consistent and every Γ' such that $\Gamma \subseteq \Gamma'$ is inconsistent.
- A logic is strongly complete if $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$. It is weakly complete if $\models \varphi$ implies $\vdash \varphi$. Strong completeness implies weak completeness, but weak completeness does not imply strong completeness.

Important facts about maximally consistent sets: Suppose that Γ is a maximally consistent set,

1. If $\vdash \varphi$ then $\varphi \in \Gamma$
2. If $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$ then $\psi \in \Gamma$
3. $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$
4. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$
5. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$

Lemma 5 (Lindenbaum's Lemma) *For each consistent set Γ, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. In other words, every consistent set Γ can be extended to a maximally consistent set.*

Definition 6 (Canonical Model) The canonical model for **K** is the model $\mathcal{M}^c = \langle W^c, R^c, V^c \rangle$ where

- $W^c = \{\Gamma \mid \Gamma \text{ is a maximally consistent set}\}$
- $\Gamma R^c \Delta$ iff $\Gamma^\Box = \{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta$
- $V^c(p) = \{\Gamma \mid p \in \Gamma\}$ $\triangleleft$

Lemma 7 (Truth Lemma) *For every $\varphi \in \mathcal{L}$, $\mathcal{M}^c, \Gamma \models \varphi$ iff $\varphi \in \Gamma$*

Theorem 8 *Every maximally consistent set Γ has a model (i.e., there is a models $\mathcal{M}$ and state w such that for all $\varphi \in \Gamma$, $\mathcal{M}, w \models \varphi$.*

Proof. Suppose that Γ is a consistent set. By Lindenbaum's Lemma, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. Then, by the Truth Lemma, for each $\varphi \in \Gamma'$, we have $\mathcal{M}^c, \Gamma' \models \varphi$. Then, in particular, every formula in Γ is true at Γ' in the canonical model. QED

Theorem 9 *If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$*

Proof. Suppose that $\Gamma \not\vdash \varphi$. Then, $\Gamma \cup \{\neg\varphi\}$ is consistent. By the above theorem, there is a model of $\Gamma \cup \{\neg\varphi\}$. Hence, $\Gamma \not\models \varphi$. QED

Suppose that **L** is a logic extending **K**. We can build a canonical model for **L** as above. The question is: Is the canonical model in the appropriate class of models?

Lemma 10 *If $\Box\varphi \to \varphi \in$ **L**, then the canonical model for **L** is reflexive.*

Proof. Suppose that $\Box\varphi \to \varphi$ is derivable in **L**. We must show that for any MCS Γ, $\Gamma R^c \Gamma$. That is, $\Gamma^\Box = \{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma$. Suppose that $\Box\psi \in \Gamma$. We must show that $\psi \in \Gamma$. This follows since $\Box\psi \to \psi \in \Gamma$ and Γ is closed under modus ponens. QED

Lemma 11 *If $\Box\varphi \to \Box\Box\varphi \in \mathbf{L}$, then the canonical model for* **L** *is transitive.*

Proof. Suppose that $\Box\varphi \to \Box\Box\varphi$ is derivable in **L**. We must show that for MCS $\Gamma, \Gamma', \Gamma''$, if $\Gamma R^c \Gamma'$ and $\Gamma' R^c \Gamma''$, then $\Gamma R^c \Gamma''$. Suppose that $\Gamma R^c \Gamma'$ and $\Gamma' R^c \Gamma''$. Then, $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma'$ and $\{\varphi \mid \Box\varphi \in \Gamma'\} \subseteq \Gamma''$. We must show $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma''$. Suppose that $\Box\psi \in \Gamma$. Then, since $\Box\psi \to \Box\Box\psi \in \Gamma$, we have $\Box\Box\psi \in \Gamma$. This means, $\Box\psi \in \Gamma'$ and $\psi \in \Gamma''$, as desired. QED

Theorem 12 **S4** *is sound and strongly complete with respect to the class of Kripke structures that are reflexive and transitive.*

Lemma 13 *If $\neg\Box\varphi \to \Box\neg\Box\varphi \in \mathbf{L}$, then the canonical model for* **L** *is Euclidean.*

Proof. Suppose that $\neg\Box\varphi \to \Box\neg\Box\varphi$ is derivable in **L**. We must show that for MCS $\Gamma, \Gamma', \Gamma''$, if $\Gamma R^c \Gamma'$ and $\Gamma R^c \Gamma''$, then $\Gamma' R^c \Gamma''$. Suppose that $\Gamma R^c \Gamma'$ and $\Gamma R^c \Gamma''$. Then, $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma'$ and $\{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Gamma''$. We must show $\{\varphi \mid \Box\varphi \in \Gamma'\} \subseteq \Gamma''$. Suppose that $\Box\psi \in \Gamma'$. If $\psi \notin \Gamma''$, then $\neg\psi \in \Gamma''$. This implies that $\Box\psi \notin \Gamma$, and hence, $\neg\Box\psi \in \Gamma$. Since $\neg\Box\psi \to \Box\neg\Box\psi \in \Gamma$, we have $\Box\neg\Box\psi \in \Gamma$. This implies that $\neg\Box\psi \in \Gamma'$, a contradiction. Hence, $\psi \in \Gamma''$, as desired. QED

Theorem 14 **S5** *is sound and strongly complete with respect to the class of Kripke structures that are equivalence relations (reflexive, transitive and symmetric).*

Completeness-via-canonicity: Let φ be a modal formula and P a property. If every normal modal logic containing φ has property P and φ is valid on any class of frames with property P, then φ is **canonical for** P.

Limitations to the above approach:

- **Undefinable Properties**: Completeness by *transforming the canonical model*: **S4** is sound and strongly complete with respect to the class of reflexive and transitive *trees*. What is the modal logic of *strict total orders*?

- **Weak Completeness**: there are normal modal logics that are not strongly complete. Eg., **KL** (**K** plus $\Box(\Box\varphi \to \varphi) \to \Box\varphi$) is not strongly complete.

- **Incompleteness** There are *consistent* normal modal logics that are not complete with respect to any class of frames (more on this later).

5 Alternative Proof of Weak Completeness

In this section we illustrate a technique for by proving weak completeness invented by Larry Moss in [1]. Since we are only interested in illustrating the technique, we focus on the smallest normal modal logic (**K**). Recall that the basic modal language is generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi$$

where p is a propositional variable (let $\mathsf{At} = \{p_1, p_2, \ldots, p_n, \ldots\}$ deonte the set of propositional variables). Define the usual boolean connectives and the modal operator $\Box$ as usual. Let $\mathcal{L}_\Diamond$ be the set of well-formed formulas.

Some notation is useful at this stage. The **height**, or **modal depth**, of a formula $\varphi \in \mathcal{L}_\Diamond$, denoted $\mathsf{ht}(\varphi)$, is longest sequence of nested modal operators. Formally, define ht as follows

$$\begin{array}{lcl} \mathsf{ht}(p_n) & = & 0 \\ \mathsf{ht}(\neg\varphi) & = & \mathsf{ht}(\varphi) \\ \mathsf{ht}(\varphi \vee \psi) & = & \max\{\mathsf{ht}(\varphi), \mathsf{ht}(\psi)\} \\ \mathsf{ht}(\Diamond\varphi) & = & 1 + \mathsf{ht}(\varphi) \end{array}$$

The **order** of a modal formula φ, written $\mathsf{ord}(\varphi)$, is the largest index of a propositional formula that appears in φ. Formally,

$$\begin{array}{lcl} \mathsf{ord}(p_n) & = & n \\ \mathsf{ord}(\neg\varphi) & = & \mathsf{ord}(\varphi) \\ \mathsf{ord}(\varphi \vee \psi) & = & \max\{\mathsf{ord}(\varphi), \mathsf{ord}(\psi)\} \\ \mathsf{ord}(\Diamond_n\varphi) & = & \mathsf{ord}(\varphi) \end{array}$$

Let $\mathcal{L}_{h,n} = \{\varphi \mid \varphi \in \mathcal{L}_\Diamond,\ \mathsf{ht}(\varphi) \leq h \text{ and } \mathsf{ord}(\varphi) \leq n\}$. Thus, for example, $\mathcal{L}_{0,n}$ is the propositional language (finite up to logical equivalence) built from the set $\{p_1, \ldots, p_n\}$ of propositional variables.

A set $T \subseteq \{p_1, \ldots, p_m\}$ corresponds to a partial valuation on At if we think of the elements of T as being true and the elements of $\{p_1, \ldots, p_m\} - T$ as being false. This partial valuation can be described by the following formula of $\mathcal{L}_{0,m}$

$$\widehat{T} = \bigwedge_{p \in T} p \wedge \bigwedge_{p \in \{p_1, \ldots, p_n\} - T} \neg p$$

Now, for each $\varphi \in \mathcal{L}_{0,m}$ it is easy to see that exactly one of the following holds: $\vdash \widehat{T} \to \varphi$ or $\vdash \widehat{T} \to \neg\varphi$. Furthermore, it is easy to show that for each $\varphi \in \mathcal{L}_{0,m}$, $\vdash \varphi \leftrightarrow \bigvee\{\widehat{T} \mid \vdash \widehat{T} \to \varphi\}$. The central idea of Moss' technique is to generalize these facts to modal logic.

It is well-known that modal logic has the *finite tree property*, i.e., when evaluating a formula φ it is enough to consider only paths of length at most the modal depth of φ. The modal generalization of the formulas described above are called **canonical sentences**. Fix a natural number n and construct a set of canonical sentences, denoted $\mathcal{C}_{h,n}$, by induction on h. Let $\mathcal{C}_{0,n} = \{\widehat{T} \mid T \subseteq \{p_1, \ldots, p_n\}\}$. Suppose that $\mathcal{C}_{h,n}$ has been defined and that $S \subseteq \mathcal{C}_{h,n}$ and $T \subseteq \{p_1, \ldots, p_n\}$. Define the formula

$$\alpha_{S,T} := \bigwedge_{\psi \in S} \Diamond\psi \wedge \Box \bigvee S \wedge \widehat{T}$$

and let $\mathcal{C}_{h+1,n} = \{\alpha_{S,T} \mid S \subseteq \mathcal{C}_{h,n},\ T \subseteq \{p_1, \ldots, p_n\}\}$. It is not hard to see that formulas of the form $\alpha_{S,T}$ play the same role in modal logic as the formulas $\widehat{T}$ in propositional logic. That is, $\alpha_{S,T}$ can be thought of as a complete description of a modal state of affairs. This is justified by the following Lemma from [1]. The proof can be found in [1] although we will repeat it here in the interest of exposition.

Lemma 15 *For any modal formula φ of modal depth at most h built from propositional variables $\{p_1, \ldots, p_n\}$ and any $\alpha_{S,T} \in \mathcal{C}_{h+1,n}$ exactly one of the following holds $\vdash \alpha_{S,T} \to \varphi$ or $\vdash \alpha_{S,T} \to \neg\varphi$.*

Proof. The proof is by induction on φ. The base case is obvious as are the boolean connectives. We consider only the modal case. Suppose that statement holds for ψ and consider the formula $\Diamond\psi$. Note that for each $\beta \in S$, the induction hypothesis applies to β and ψ. Thus for each $\beta \in S$, either $\vdash \beta \to \psi$ or $\vdash \beta \to \neg\psi$. There are two cases: 1. there is some $\beta \in S$ such that $\vdash \beta \to \psi$ and 2. for each $\beta \in S$, $\vdash \beta \to \neg\psi$. Suppose case 1 holds and $\beta \in S$ is such that $\vdash \beta \to \psi$. Then, it is easy to show that in **K**, $\vdash \Diamond\beta \to \Diamond\psi$. Hence, by construction of $\alpha_{S,T}$, $\vdash \alpha_{S,T} \to \Diamond\psi$. Suppose we are in the second case. Using propositional reasoning, $\vdash \bigvee S \to \neg\psi$. Then, $\vdash \Box \bigvee S \to \Box\neg\psi$. Hence, by construction of $\alpha_{S,T}$, $\vdash \alpha_{S,T} \to \neg\Diamond\psi$. QED

This lemma demonstrates that we can think of these formulas as complete descriptions of a state (up to finite depth) in some Kripke structure. There are a few other facts that are relevant at this point. The proofs can be found in [1] and we will not repeat them here. Given a set of formulas X, let $\bigoplus X$ denote *exactly one of* X. Formally, if $X = \{\varphi_1, \ldots, \varphi_n\}$, then $\bigoplus X$ is short for $\bigvee_{i=1,\ldots,n}(\varphi_i \wedge \neg \bigvee_{j \neq i} \varphi_j)$.

Lemma 16 1. *For any h, $\vdash \bigoplus \mathcal{C}_{h,n}$ (and hence $\vdash \bigvee \mathcal{C}_{h,n}$)*

2. *For any formula φ of height h, $\vdash \varphi \leftrightarrow \bigvee\{\alpha \mid \alpha \in \mathcal{C}_{h,n},\ \vdash \alpha \to \varphi\}$*

Moss constructs a (finite) Kripke model from the set of formulas $\mathcal{C}_{h,n}$ as follows. Let $\mathbb{C}_{h,n} = \langle \mathcal{C}, R, V \rangle$ where

1. $\mathcal{C} \subseteq \mathcal{C}_{h,n}$ is the set of all **K-consistent** formulas from $\mathcal{C}_{h,n}$

2. For $\alpha, \beta \in \mathcal{C}$, $\alpha R \beta$ provided $\alpha \wedge \Diamond\beta$ is consistent

3. for $p \in \{p_1, \ldots, p_n\}$, $V(p) = \{\alpha \mid \alpha \in \mathcal{C}, \vdash \alpha \to p\}$.

The truth Lemma connects truth of φ at a state α and the derivability of the implication $\alpha \to \varphi$. We first need an existence Lemma whose proof can be found in [1]

Lemma 17 (Existence Lemma, [1]) *Suppose that $\varphi \in \mathcal{L}_{h,n}$ and $\mathbb{C}_{h,n} = \langle \mathcal{C}, R, V\rangle$ is as defined above. If $\alpha \wedge \Diamond\varphi$ is* **K***-consistent then there is a $\beta \in \mathcal{C}$ such that $\alpha \wedge \Diamond\beta$ is* **K***-consistent and $\vdash \beta \to \varphi$.*

The proof uses Lemma 16 and can be found in [1].

Lemma 18 (Truth Lemma, [1]) *Suppose that $\varphi \in \mathcal{L}_{h,n}$ and $\mathbb{C}_{h,n} = \langle \mathcal{C}, R, V\rangle$ is as defined above. Then for each $\alpha \in \mathcal{C}$, $\mathbb{C}_{h,n}, \alpha \models \varphi$ iff $\vdash_{\mathbf{K}} \alpha \to \varphi$.*

Proof. As usual, the proof is by induction on φ. The base case and boolean connectives are straightforward. The only interesting case is the modal operator. Suppose that $\mathbb{C}_{h,n}, \alpha \models \Diamond\psi$. Then there is some $\beta \in \mathcal{C}$ such that $\alpha R \beta$ and $\mathbb{C}_{h,n}, \beta \models \psi$. By the definition of R, $\alpha \wedge \Diamond\beta$ is **K**-consistent. By Lemma 15, either $\vdash \alpha \to \Diamond\psi$ or $\vdash \alpha \to \neg\Diamond\psi$. If $\vdash \alpha \to \Diamond\psi$ we are done. Suppose that $\vdash \alpha \to \neg\Diamond\psi$. Now, by the induction hypothesis, $\vdash \beta \to \psi$. Hence $\vdash \Diamond\beta \to \Diamond\psi$. But this contradicts the assumption that $\alpha \wedge \Diamond\beta$ is **K**-consistent. Suppose that $\vdash \alpha \to \Diamond\psi$. Then $\alpha \wedge \Diamond\psi$ is **K**-consistent. Hence by Lemma 17, there is a $\beta \in \mathcal{C}$ such that $\alpha \wedge \Diamond\beta$ is **K**-consistent and $\vdash \beta \to \psi$. But this means that $\mathbb{C}_{h,n}, \alpha \models \Diamond\psi$. QED

The weak completeness theorem easily follows from the above Lemmas.

Theorem 19 **K** *is weakly complete, i.e., for each $\varphi \in \mathcal{L}_\Diamond$, if $\models \varphi$, then $\vdash_{\mathbf{K}} \varphi$.*

Proof. Let h and n be large enough so that $\varphi \in \mathcal{L}_{h,n}$ and suppose that $\models \varphi$. Then, in particular, φ is valid in $\mathbb{C}_{h,n}$. Thus for each $\alpha \in \mathcal{C}$, $\mathbb{C}_{h,n}, \alpha \models \varphi$. Hence by Lemma 18, for each $\alpha \in \mathcal{C}$, $\vdash \alpha \to \varphi$. Hence, $\vdash \bigvee \mathcal{C} \to \varphi$. By Lemma 16, $\vdash \bigvee \mathcal{C}$. Therefore, $\vdash \varphi$. QED

In [1], Moss uses the above technique to show that a number of well-known modal logics are weakly complete.

References

[1] Larry Moss Finite models constructed from canonical formulas. Journal of Philosophical Logic, 36:6, pp. 605 - 640, 2005.