Lecture 2: Expressivity and Invariance

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1 Propositional Modal Logic

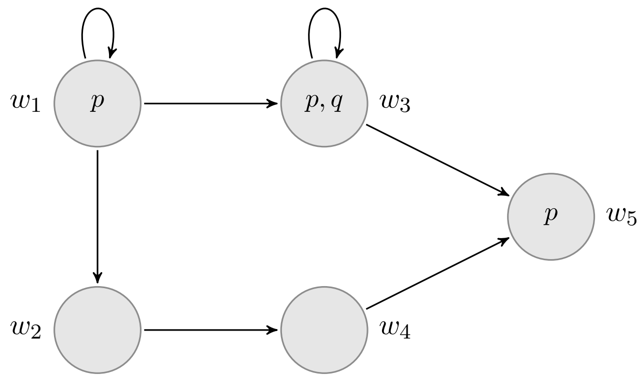
- **Language**: $p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\psi$, $p \in \mathsf{At}$ (atomic propositions), Boolean connectives defined as usual, $\Box\varphi := \neg\Diamond\neg\varphi$
- **Frame**: $\langle W, R\rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$
- **Model**: $\langle W, R, V\rangle$, where $\langle W, R\rangle$ is a frame and $V : \mathsf{At} \to \wp(W)$ (Kripke structure)
- **Truth at a state in a model**: $\mathcal{M}, w \models \varphi$
  - $\mathcal{M}, w \models p$ iff $w \in V(p)$
  - $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
  - $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
  - $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$

  Since $\Box\varphi$ is defined to be $\neg\Diamond\neg\varphi$, we have

  - $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
- **Validity**: Suppose that $\mathcal{F} = \langle W, R\rangle$ is a frame and $\mathcal{M} = \langle W, R, V\rangle$ is a model.
  - φ is satisfiable when there is a model $\mathcal{M} = \langle W, R, V\rangle$ with a state $w \in W$ such that $\mathcal{M}, w \models \varphi$
  - Valid on a model, $\mathcal{M} \models \varphi$: for all $w \in W$, $\mathcal{M}, w \models \varphi$
  - Valid on a frame, $\mathcal{F} \models \varphi$: for all $\mathcal{M}$ based on $\mathcal{F}$, for all $w \in W$, $\mathcal{M}, w \models \varphi$
    for all functions V, for all $w \in W$, $\langle W, R, V\rangle, w \models \varphi$
  - Valid at a state on a frame at a state $w \in W$, $\mathcal{F}, w \models \varphi$: for all $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M}, w \models \varphi$
  - Valid in a class F of frames, $\models_{\mathsf{F}} \varphi$: for all $\mathcal{F} \in \mathsf{F}$, $\mathcal{F} \models \varphi$

2 Tutorial Questions

- Consider the following model:



Determine which of the following formulas are true at w_1 (explain your answer)

1. $\Diamond(q \wedge \Diamond q)$
2. $\Box\bot$
3. $\Box\Diamond\Diamond p$
4. $\Box\Box\Box p$

- Determine which of the following formulas are valid on the above model (explain your answer)

1. $\Diamond\Diamond\Box\bot$
2. $q \to \Diamond q$
3. $\Diamond\Box p \vee \Box\Diamond p$

- Let $\mathcal{F} = \langle B, R_1, R_2 \rangle$ be a frame where B is the set of all finite strings of 0s and 1s, and the relations R_1 and R_2 are defined by:

  sR_1t iff $t = s0$ or $t = s1$

  sR_2t iff t is a proper initial segment of s.

  Which of the following formulas are valid on this frame?

  1. $(\Diamond_1 p \wedge \Diamond_1 q) \rightarrow \Diamond_1(p \wedge q)$

  2. $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \rightarrow (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$

  3. $(\Diamond_2 p \wedge \Diamond_2 q \wedge \Diamond_2 r) \rightarrow (\Diamond_2(p \wedge q) \vee \Diamond_2(p \wedge r) \vee \Diamond_2(q \wedge r))$

- Find a model with a state that makes $p \rightarrow \Diamond p$ false. Show that if the frame is reflexive, then $p \rightarrow \Diamond p$ is valid.

- Find a model with a state that makes $\Diamond\Diamond p \rightarrow \Diamond p$ false. Show that if the frame is transitive, then $\Diamond\Diamond p \rightarrow \Diamond p$ is valid.

3 Expressivity and Invariance

Consider the following modalities:

- $\mathcal{M}, w \models A\varphi$ iff for all $w \in W$, $\mathcal{M}, w \models \varphi$
- $\mathcal{M}, w \models \Diamond^{\leftarrow}\varphi$ iff there is a $v \in W$, vRw and $\mathcal{M}, v \models \varphi$.
- $\mathcal{M}, w \models \Diamond_n\varphi$ iff there are $v_1, \ldots, v_n$ such that for all $1 \leq j \neq k \leq n$, $v_j \neq v_k$, for all $j = 1, \ldots, n$, wRv_j and for all $j = 1, \ldots, n$, $\mathcal{M}, v_j \models \varphi$.

  For instance, $\Diamond_2\varphi$ is true at a state if there are at least two accessible states that satisfy φ.
- $\mathcal{M}, w \models \circlearrowleft$ iff wRw

Are these modalities definable using the basic modal language? Intuitively, the answer is "no", but how do we *prove* this?

Model Constructions

- **Disjoint Union**: Let $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$. The disjoint union is the structure $\mathcal{M}_1 \uplus \mathcal{M}_2 = \langle W, R, V\rangle$ where
  - $W = W_1 \cup W_2$ (disjoint union)
  - $R = R_1 \cup R_2$
  - for all $p \in \mathsf{At}$, $V(p) = V_1(p) \cup V_2(p)$

  Lemma For each collection of Kripke structures $\{\mathcal{M}_i \mid i \in I\}$, for each $w \in W_i$, $\mathcal{M}_i, w \models \varphi$ iff $\biguplus_{i\in I} \mathcal{M}_i, w \models \varphi$

  Fact The universal modality is not definable in the basic modal language.
- **Generated Submodel**: $\mathcal{M}' = \langle W', R', V'\rangle$ is a generated submodel of $\mathcal{M} = \langle W, R, V\rangle$ provided
  - $W' \subseteq W$ is R-closed:
    for each $w' \in W$ and $v \in W$, if wRv then $v \in W'$.
  - $R' = R \cap W' \times W'$
  - for all $p \in \mathsf{At}$, $V'(p) = V(p) \cap W'$

  Lemma If $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$ then for each $w \in W'$, $\mathcal{M}', w \models \varphi$ iff $\mathcal{M}, w \models \varphi$

  Fact The backwards looking modality is not definable in the basic modal language.
- **Bounded Morphism** A bounded morphism between models $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ is a function f with domain W and range W' such that:

  Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $f(w) \in V'(p)$
  Morphism: if wRv then $f(w)Rf(v)$
  Zag: if $f(w)R'v'$ then $\exists v \in W$ such that $f(v) = v'$ and wRv

Lemma If $\mathcal{M}'$ is a bounded morphic image of $\mathcal{M}$ then for each $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', f(w) \models \varphi$

Fact Counting modalities are not definable in the basic modal language (eg., $\Diamond_1\varphi$ iff φ is true in more than 1 accessible world).

- **Tree Unfoldings**: The unfolding of $\mathcal{M} = \langle W, R, V\rangle$ with root w is $\vec{\mathcal{M}} = \langle \vec{W}, \vec{R}, \vec{V}\rangle$, where $\vec{W}$ is the set of paths starting at w, $(w, \ldots, w_n)\ \vec{R}\ (w, \ldots, w_n, w_{n+1})$ iff $w_n R w_{n+1}$ and $(w, \ldots, w_n) \in V(p)$ iff $w_n \in V(p)$.

  Lemma. Every satisfiable modal formula is satisfiable at the root of a tree.

- **Bisimulation**: A bisimulation between $\mathcal{M} = \langle W, R, V\rangle$and $\mathcal{M}' = \langle W', R', V'\rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

  Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$
  Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
  Zag: if $w'R'v'$ then $\exists v \in W$ such that vZv' and wRv

  - *We write $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ if there is a Z such that wZw'.*
  - *We write $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ iff for all $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.*
  - **Lemma** If $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$.
  - **Lemma** On finite models, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.
  - **Lemma** On m-**saturated models**, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

Proposition. Any Kripke structure is the bounded morphic image of a disjoint union of rooted Kripke structures (in fact, tree structures).

Defining classes of models/frames

- $PKS(\varphi) = \{(\mathcal{M}, w) \mid \mathcal{M}, w \models \varphi\}$
- $KS(\varphi) = \{\mathcal{M} \mid \mathcal{M} \models \varphi\}$
- $PFR(\varphi) = \{(\mathcal{F}, w) \mid (\mathcal{F}, V), w \models \varphi$ for all valuations $V\}$
- $FR(\varphi) = \{\mathcal{F} \mid (\mathcal{F}, V), w \models \varphi$ for all $w \in dom(\mathcal{M})$ and valuations $V\}$

Advanced Topic: Ultrafilter extensions

Fact. Closure under generated subframe, bounded morphic images, and disjoint unions is not sufficient to guarantee definability by a modal formula for a class of frames. (eg., frames defined by $\forall x \exists y(xRy \wedge yRy)$).

- **Ultrafilter Extensions**: Let $m(X) = \{w \mid \text{there is a } v \text{ such that } wRv \text{ and } v \in X\}$ and $l(X) = \overline{m(\overline{X})} = \{w \mid \text{ for all } v, \text{ if } wRv \text{ then } v \in X\}$. An ultrafilter extension is a model

$$ue(\mathcal{M}) = \langle Uf(W), R^{ue}, V^{ue} \rangle$$

where $Uf(W) = \{u \mid u \text{ is an ultrafilter over } W\}$, $uR^{ue}u'$ iff for all $X \subseteq W$, if $X \in u'$ then $m(X) \in u$, and $V(p) = \{u \mid V(p) \in u\}$.

Fact. For all models $\mathcal{M}$, $w \leftrightsquigarrow u_w$, where u_w is the principle ultrafilter generated by w.
Fact. For all models $\mathcal{M}$, $ue(\mathcal{M})$ is m-saturated.
Fact. $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ iff $ue(\mathcal{M}), u_w \underline{\leftrightarrow} ue(\mathcal{M}'), u_{w'}$