Relational Semantics for Modal Logic*

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These notes are a very brief introduction to relational semantics for modal logic. The goal is to provide just enough details to motivate the discussion of neighborhood semantics and facilitate a comparison between the two semantics. There are many textbooks that you can consult for more information. The following is a list of some useful texts (this is not a complete list, but a pointer to books that covers topics related to issues discussed in this book). [1]

- *Modal Logic for Open Minds* (2010) by Johan van Benthem. An introductory textbook on modal logic that is focused on the underlying theory and main philosophical and mathematical applications.
- *Modal Logic* (2001) by Patrick Blackburn, Maarten de Rijke and Yde Venema. An advanced, but very accessible, textbook foucsed on the main technical results about propositional modal logic.
- *Modal Logic* (1980) by Brian Chellas. An introduction to modal logic that covers both normal and non-normal systems.
- *First Order Modal Logic* (1999) by Melvin Fitting and Richard Mendelsohn. This book provides both a philosophical and technical introduction to first-order modal logic.

*These notes are an extended version of the Appendix from my book *Neighborhood Semantics for Modal Logic* (Pacuit, 2017).

[1] This is not a complete list, but a pointer to books that covers topics related to issues discussed in this book. See Chagrov and Zakharyaschev (1997); Kracht (1999); Goldblatt (1992); and Humberstone (2016) for different perspectives on modal logic.

1 Language and Semantics

Definition 1 (The Basic Modal Language) Suppose that $\mathsf{At} = \{p, q, r, \ldots\}$ is a (finite or countable) set of sentence letters, or atomic propositions. The set of well-formed formulas generated from At, denoted $\mathcal{L}(\mathsf{At})$, is the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi \mid \Diamond\varphi$$

where $p \in \mathsf{At}$. ◁

Additional propositional connectives (e.g., $\vee, \rightarrow, \leftrightarrow$) are defined as usual. It will be convenient to introduce special formulas '$\top$' and '$\bot$', meaning 'true' and 'false', respectively. Typically, $\bot$ is defined to be $p \wedge \neg p$ (where $p \in \mathsf{At}$) and $\top$ is $\neg\bot$. If the set of atomic propositions is empty, then add $\bot$ and $\top$ to the language. Examples of modal formulas include: [2] $\Box\bot$, $\Box\Diamond\top$, $p \rightarrow \Box(q \wedge r)$, and $\Box(p \rightarrow (q \vee \Diamond r))$. To simplify the notation, I write $\mathcal{L}$ for $\mathcal{L}(\mathsf{At})$ when the set of atomic propositions At is understood.

Remark 2 (Modal Operators) According to Definition 1, $\mathcal{L}$ contains two *unary* modal operators. In this text, I will discuss languages that contain more than two unary modalities and languages that contain modalities of other arities (e.g., the binary modality in Section **??**). Furthermore, it is often convenient to *define* $\Diamond\varphi$ as $\neg\Box\neg\varphi$ (cf. Lemma **??**).

One language, many readings. There are many possible readings for the modal operators '$\Box$' and '$\Diamond$'. Here are some samples:

- **Alethic Reading**: $\Box\varphi$ means 'φ is necessary' and $\Diamond\varphi$ means 'φ is possible'.
- **Deontic Reading**: $\Box\varphi$ means 'φ is obligatory' and $\Diamond\varphi$ means 'φ is permitted'. In this literature, 'O' typically is used instead of '$\Box$' and 'P' instead of '$\Diamond$'.
- **Epistemic Reading**: $\Box\varphi$ means 'φ is known' and $\Diamond\varphi$ means 'φ is consistent with the knower's current information'. In this literature, 'K' typically is used instead of '$\Box$' and 'L' instead of '$\Diamond$'.
- **Temporal Reading**: $\Box\varphi$ means 'φ will always be true' and $\Diamond\varphi$ means 'φ will be true at some point in the future'. In this literature, 'G' typically is used instead of '$\Box$ and 'F' instead of '$\Diamond$'.

I conclude this brief introduction to the basic modal language with the standard definition of a substitution between formulas.

Definition 3 (Substitution) A **substitution** σ is a function from atomic propositions to well-formed formulas: $\sigma : \mathsf{At} \rightarrow \mathcal{L}(\mathsf{At})$. A substitution σ is extended to a function on all formulas, denoted $\overline{\sigma} : \mathcal{L}(\mathsf{At}) \rightarrow \mathcal{L}(\mathsf{At})$, by recursion on the structure of the formulas:

[2] To simplify the presentation, I will typically drop the outermost parentheses.

1. $\overline{\sigma}(p) = \sigma(p)$
2. $\overline{\sigma}(\varphi \wedge \psi) = \overline{\sigma}(\varphi) \wedge \overline{\sigma}(\psi)$
3. $\overline{\sigma}(\Box\varphi) = \Box\overline{\sigma}(\varphi)$
4. $\overline{\sigma}(\Diamond\varphi) = \Diamond\overline{\sigma}(\varphi)$

For simplicity, I will often identify σ and $\overline{\sigma}$ and write φ^σ for $\sigma(\varphi)$. ◁

For example, if $\sigma(p) = \Box\Diamond(p \wedge q)$ and $\sigma(q) = p \wedge \Box q$, then

$$(\Box(p \wedge q) \to \Box p)^\sigma = \Box((\Box\Diamond(p \wedge q)) \wedge (p \wedge \Box q)) \to \Box(\Box\Diamond(p \wedge q)).$$

Exercise 1

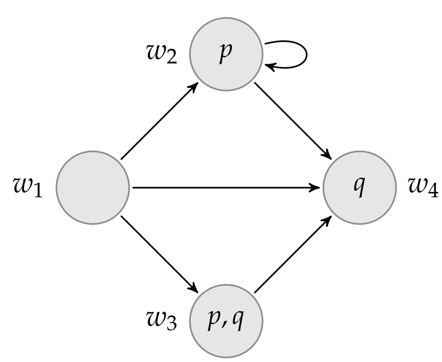
1. *Suppose that $\sigma(p) = \Box q$ and $\sigma(q) = (p \to \Box q)$. Find $(\Box(p \to q) \to (\Box p \to \Box q))^\sigma$.*
2. *Suppose that $\sigma(p) = \neg p$. Find $(\Box p \leftrightarrow \neg\Diamond\neg p)^\sigma$.*
3. *Show that $\varphi^\sigma = \varphi$ iff $\sigma(p) = p$ for all atomic propositions p occurring in φ.*

Definition 4 (Relational Frame and Model) A **relational frame** is a tuple $\langle W, R\rangle$ where W is a nonempty set (elements of W are called **states**), $R \subseteq W \times W$ is a relation on W. A **relational model** (also called a Kripke model) is a triple $\mathcal{M} = \langle W, R, V\rangle$ where $\langle W, R\rangle$ is a relational frame and $V : \mathsf{At} \to \wp(W)$ is a **valuation function** assigning sets of states to atomic propositions. ◁

Example 5 *The following picture represents the relational structure $\mathcal{M} = \langle W, R, V\rangle$ where $W = \{w_1, w_2, w_3, w_4\}$,*

$$R = \{(w_1, w_2), (w_1, w_3), (w_1, w_4), (w_2, w_2), (w_2, w_4), (w_3, w_4)\}$$

and $V(p) = \{w_2, w_3\}$ and $V(q) = \{w_3, w_4\}$.



Formulas of $\mathcal{L}$ are interpreted at states in a relational model.

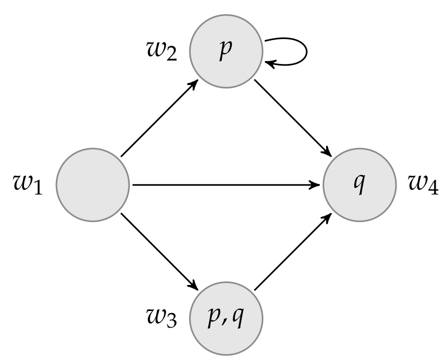
Definition 6 (Truth of Modal Formulas) Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a relational model. Truth of a modal formula $\varphi \in \mathcal{L}(\mathsf{At})$ at a state w in $\mathcal{M}$, denoted $\mathcal{M}, w \models \varphi$, is defined inductively as follows:

1. $\mathcal{M}, w \models p$ iff $w \in V(p)$ (where $p \in \mathsf{At}$)
2. $\mathcal{M}, w \models \top$ and $\mathcal{M}, w \not\models \bot$
3. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
4. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
5. $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
6. $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$ $\triangleleft$

Two remarks about this definition. First, note that truth for the other boolean connectives ($\rightarrow, \vee, \leftrightarrow$) is not given in the above definition. This is not necessary since these connectives are *definable* from '$\neg$' and '$\wedge$'. [3] As an exercise, make sure you can specify the truth definition in the style of the definition above for each of the boolean connectives not mentioned. Second, note the analogy between '$\Box$' and the universal quantifier and '$\Diamond$' and the existential quantifier.

Remark 7 (Truth Set) Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a relational model. For each $\varphi \in \mathcal{L}$, let $[\![\varphi]\!]_{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \varphi\}$ be the **truth set** of φ (in $\mathcal{M}$). Formally, we can adapt the clauses discussed in Section **??** to define a function $[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \rightarrow \wp(W)$ (recall that $\wp(W) = \{X \mid X \subseteq W\}$ is the powerset of W).

Example 8 *To illustrate the above definition of truth of modal formula, recall the relational model from Example 5:*



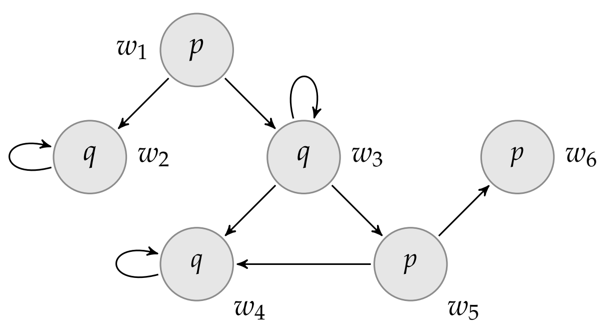
- $\mathcal{M}, w_3, \models \Box q$: *$w_4$ is the only worlds accessible from w_3 and q is true at w_4.*

[3] For example, $\varphi \rightarrow \psi$ can be defined as (i.e., is logically equivalent to) $\neg(\varphi \wedge \neg\psi)$.

- $\mathcal{M}, w_1 \models \Diamond q$: *there is a state accessible from w_1 (namely w_3) where q is true.*
- $\mathcal{M}, w_1 \models \Diamond\Box q$: *$w_3$ is accessible from w_1 and q is true in all of the worlds accessible from w_3.*
- $\mathcal{M}, w_4 \models \Box\bot$: *there are no worlds accessible from w_4, so any formula beginning with '$\Box$' will be true (this is analogous to the fact the universal sentences are true in any first-order structure where the domain is empty). Similarly, any formula beginning with a '$\Diamond$' will be false (again, this is analogous to the fact that existential statements are false in first-order structures with empty domains).* ◁

For an extended discussion surrounding the interpretation modal formulas in relational models, see Chapter 2 of (van Benthem, 2010).

Exercise 2 *Consider the following relational model.*



1. $\Box q \to \Box\Box q$
2. $\Box\Box q \to \Box q$
3. $\Diamond(\Diamond q \wedge \Diamond p)$
4. $\Diamond\Box\bot$
5. $\Box(\Box q \to q) \to \Box q$

For each formula to the right, list the states where the formula is true.

Exercise 3 *Consult* `http://pacuit.org/modal/tutorial/` *for more examples to test your understanding of the definition of truth for modal formulas over relational models.*

2 Validity

Definition 9 (Validity) A modal formula $\varphi \in \mathcal{L}$ is **valid in a relational model** $\mathcal{M} = \langle W, R, V\rangle$, denoted $\mathcal{M} \models \varphi$, provided $\mathcal{M}, w \models \varphi$ for each $w \in W$. Suppose that $\mathcal{F} = \langle W, R\rangle$ is a relational frame. A modal formula $\varphi \in \mathcal{L}$ is valid on $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, provided $\mathcal{M} \models \varphi$ for all models based on $\mathcal{F}$ (i.e., all models $\mathcal{M} = \langle \mathcal{F}, V\rangle$). Suppose that F is a class of relational frames. A modal formula φ is **valid on** F, denoted $\models_{\mathsf{F}} \varphi$, provided $\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathsf{F}$. If F is the class of all relational frames, then I will write $\models \varphi$ instead of $\models_{\mathsf{F}} \varphi$. ◁

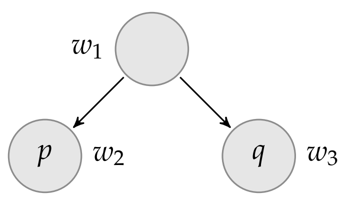
In order to show that a modal formula φ is valid, it is enough to argue informally that φ is true at an arbitrary state in an arbitrary relational model. On the other hand, to show a modal formula φ is not valid, one must provide a counter example (i.e., a relational model and state where φ is false).

Fact 10 $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ *is valid.*

Proof. Suppose $\mathcal{M} = \langle W, R, V\rangle$ is an arbitrary relational model and $w \in W$ an arbitrary state. We will show $\mathcal{M}, w \models \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$. Suppose that $\mathcal{M}, w \models \Box\varphi \wedge \Box\psi$. Then $\mathcal{M}, w \models \Box\varphi$ and $\mathcal{M}, w \models \Box\psi$. Suppose that $v \in W$ and wRv. Then $\mathcal{M}, v \models \varphi$ and $\mathcal{M}, v \models \psi$. Hence, $\mathcal{M}, v \models \varphi \wedge \psi$. Since v is an arbitrary state accessible from w, we have $\mathcal{M}, w \models \Box(\varphi \wedge \psi)$. QED

Fact 11 $(\Diamond\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \psi)$ *is not valid.*

Proof. We must find a relational model that has a state where an instance of $(\Diamond\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \psi)$ is false. Consider the following instance of the above formula: $(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$, and let $\mathcal{M} = \langle W, R, V\rangle$ be the following relational model:



We have that $\mathcal{M}, w_1 \models \Diamond p \wedge \Diamond q$ (why?), but $\mathcal{M}, w_1 \not\models \Diamond(p \wedge q)$ (why?). Hence, $\mathcal{M}, w_1 \not\models (\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$. QED

Exercise 4 *Determine which of the following formulas are valid (prove your answers):*

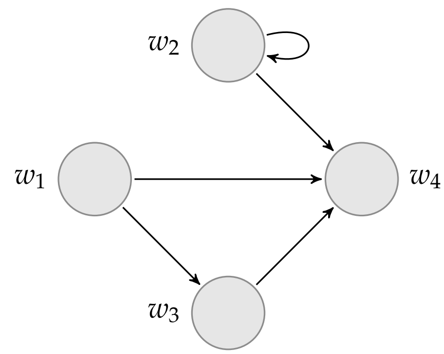
1. $\Box\varphi \to \Diamond\varphi$
2. $\Box(\varphi \vee \neg\varphi)$
3. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
4. $\Box\varphi \to \varphi$
5. $\varphi \to \Box\Diamond\varphi$
6. $\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$

3 Definability

Remark 7 explains how to assign to every modal formula $\varphi \in \mathcal{L}$ a set of states in a relational model $\mathcal{M} = \langle W, R, V\rangle$ (i.e., the truth set of φ, denoted $[\![\varphi]\!]_{\mathcal{M}}$). It is natural to ask about the converse: Given and arbitrary set, when does a formula uniquely pick out that set?

Definition 12 (Definable Subsets) Let $\mathcal{M} = \langle W, R, V\rangle$ be a relational model. A set $X \subseteq W$ is **definable in** $\mathcal{M}$ provided $X = [\![\varphi]\!]_{\mathcal{M}}$ for some modal formula $\varphi \in \mathcal{L}$. ◁

Example 13 *All four of the states in the relational model below are uniquely defined by a modal formula:*



- *$\{w_4\}$ is defined by* $\Box\bot$
  (w_4 is the only "dead-end" state*)*
- *$\{w_3\}$ is defined by* $\Diamond\Box\bot \wedge \Box\Box\bot$
  (w_3 can only see a "dead-end" state*)*
- *$\{w_2\}$ is defined by* $\Diamond\Diamond\Diamond\top$
  (w_2 is the only state where 3 steps can be taken*)*
- *$\{w_1\}$ is defined by* $\Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)$
  (w_1 is the only state that can see w_3*)*

Given the above observations, it is not hard to see that all *subsets of $W = \{w_1, w_2, w_3, w_4\}$ are definable (why?). However, note that even in finite relational models, not all subsets may be definable. A problem can arise if states cannot be distinguished by modal formulas. For example, if the reflexive arrow is dropped in the relational model above, then w_2 and w_3* cannot *be distinguished by a modal formula (there are ways to formally prove this, but see if you can informally argue why w_2 and w_3 cannot be distinguished).*

The next two definitions make precise what it means for two states to be *indistinguishable* by a modal formula.

Definition 14 (Modal Equivalence) Let $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ be two relational models. We say $\mathcal{M}_1, w_2$ and $\mathcal{M}_2, w_2$ are **modally equivalent** provided

$$\text{for all modal formulas } \varphi \in \mathcal{L},\ \mathcal{M}_1, w_1 \models \varphi \text{ iff } \mathcal{M}_2, w_2 \models \varphi$$

We write $\mathcal{M}_1, w_1 \leftrightsquigarrow \mathcal{M}_2, w_2$ if $\mathcal{M}_1, w_1$ and $\mathcal{M}_2, w_2$ are modally equivalent. (Note that it is assumed $w_1 \in W_1$ and $w_2 \in W_2$) $\lhd$

Definition 15 (Bisimulation) Let $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ be two relational models. A nonempty relation $Z \subseteq W_1 \times W_2$ is called a **bisimulation** provided for all $w_1 \in W_1$ and $w_2 \in W_2$, if $w_1 Z w_2$ then

1. (atomic harmony) For all $p \in \mathsf{At}$, $w_1 \in V_1(p)$ iff $w_2 \in V_2(p)$.
2. (zig) If $w_1 R_1 v_1$ then there is a $v_2 \in W_2$ such that $w_2 R_2 v_2$ and $v_1 Z v_2$.
3. (zag) If $w_2 R_2 v_2$ then there is a $v_1 \in W_1$ such that $w_1 R_1 v_1$ and $v_1 Z v_2$.

We write $\mathcal{M}_1, w_1 \underline{\leftrightarrow} \mathcal{M}_2, w_2$ if there is a bisimulation relating w_1 with w_2. $\lhd$

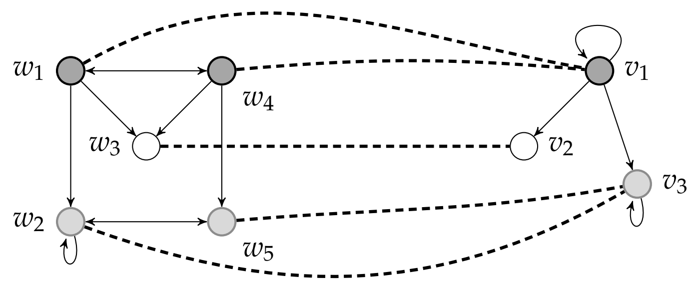
Definition 14 and 15 provide two concrete ways to answer the question: *when are two states the same*? The following is a very useful (and instructive!).

Exercise 5 *1. Prove that $\leftrightsquigarrow$ and $\underline{\leftrightarrow}$ are equivalence relations.*

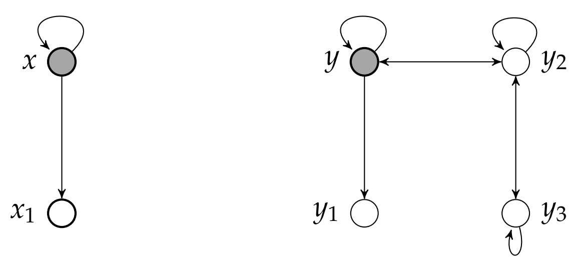
2. *Prove that if X is a definable subset of $\mathcal{M} = \langle W, R, V \rangle$, then X is closed under the $\leftrightsquigarrow$ relation (if $w \in X$ and $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}, v$ then $v \in X$).*

3. *Prove that there is a largest bisimulation: given $\{Z_i \mid i \in I\}$ a set of bisimulations relating the relational models $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2 \rangle$ (i.e., for each $i \in I$, $Z_i \subseteq W_1 \times W_2$ satisfies Definition 15), show that the relation $Z = \bigcup_{i \in I} Z_i$ is a bisimulation.*

Example 16 (Bisimulation Example) The dashed lines is a bisimulation between the following two relational models (for simplicity, we do assume that all atomic propositions are false):



On the other hand, there is no bisimulation relating the states x and y in the following two relational models:



Using Lemma 17 below, we can *prove* that there is no bisimulation relating x and y. We first note that $\Box(\Diamond\Box\bot \vee \Box\bot)$ is true at state x but not true at state y. Then by Lemma 17, x and y cannot be bisimilar.

Lemma 17 (Modal Invariance Lemma) *Suppose $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2 \rangle$ are relational models. For all $w \in W_1$ and $v \in W_2$, if $\mathcal{M}_1, w \underline{\leftrightarrow} \mathcal{M}_2, v$ then $\mathcal{M}_1, w \leftrightsquigarrow \mathcal{M}_2, v$.*

Proof. Suppose that $\mathcal{M}_1, w \underline{\leftrightarrow} \mathcal{M}_2, v$. Then, there is a bisimulation Z such that wZv. The proof is by induction on the structure of φ. The base case is when φ is p, an atomic proposition. By the atomic harmony condition, since wZv, we have $V_1(w, p) = V_2(v, p)$. Hence, $\mathcal{M}_1, w \models p$ iff $\mathcal{M}_2, v \models p$. There are three cases to consider:

<u>Case 1</u>: φ is $\psi_1 \wedge \psi_2$. Then,

$$\begin{array}{rlll} \mathcal{M}_1, w \models \psi_1 \wedge \psi_2 & \text{iff} & \mathcal{M}_1, w \models \psi_1 \text{ and } \mathcal{M}_1, w \models \psi_2 & \text{(Def. of Truth)} \\ & \text{iff} & \mathcal{M}_2, v \models \psi_1 \text{ and } \mathcal{M}_2, v \models \psi_2 & \text{(Induction hypothesis)} \\ & \text{iff} & \mathcal{M}_2, v \models \psi_1 \wedge \psi_2 & \text{(Def. of truth)} \end{array}$$

Case 2: φ is $\neg\psi$. Then,

$$\begin{array}{rlll} \mathcal{M}_1, w \models \neg\psi & \text{iff} & \mathcal{M}_1, w \not\models \psi & \text{(Def. of Truth)} \\ & \text{iff} & \mathcal{M}_2, v \not\models \psi & \text{(Induction hypothesis)} \\ & \text{iff} & \mathcal{M}_2, v \models \neg\psi & \text{(Def. of truth)} \end{array}$$

Case 3: φ is $\Box\psi$. Suppose that $\mathcal{M}_1, w \models \Box\psi$. Then for each w', if wR_1w', then $\mathcal{M}_1, w' \models \psi$. We will show that $\mathcal{M}_2, v \models \Box\psi$. Let v' be any state in W_2 with vR_2v'. By the zig condition, there is a $w' \in W_1$ such that wR_1w' and $w'Zv'$. Since $\mathcal{M}_1, w \models \Box\psi$ and wR_1w', we have $\mathcal{M}_1, w' \models \psi$. By the induction hypothesis, $\mathcal{M}_2, v' \models \psi$. Since v' is an arbitrary state with vR_2v', we have $\mathcal{M}_2, v \models \Box\psi$. The converse direction is similar (it makes use of the zag condition). QED

Lemma 18 *Suppose $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ are* finite *relational models. If $\mathcal{M}_1, w_1 \leftrightsquigarrow \mathcal{M}_2, w_2$ then $\mathcal{M}_1, w_1 \underline{\leftrightarrow} \mathcal{M}_2, w_2$.*

Proof. We show that $\leftrightsquigarrow$ is a bisimulation. The atomic harmony condition is obvious. We prove the zag condition. Suppose that $\mathcal{M}_1, w_1 \leftrightsquigarrow \mathcal{M}_2, w_2$, $w_2R_2v_2$, but there is no v_1 such that $w_1R_1v_1$ and $\mathcal{M}_1, v_1 \leftrightsquigarrow \mathcal{M}_2, v_2$. Note that there are only finitely many states that are accessible from w_1. That is, $\{w \mid w_1R_1w\}$ is a finite set. Suppose that $\{w \mid w_1R_1w\} = \{w^1, w^2, \ldots, w^m\}$. By assumption, for each w^i we have $\mathcal{M}_1, w^i \not\leftrightsquigarrow \mathcal{M}_2, v_2$. Hence, for each w^i, there is a formula φ_i such that $\mathcal{M}_1, w^i \not\models \varphi_i$ but $\mathcal{M}_2, v_2 \models \varphi_i$. Then, $\mathcal{M}_2, v_2 \models \bigwedge_{i=1,\ldots,m} \varphi_i$. Since $w_2R_2v_2$, we have $\mathcal{M}_2, w_2 \models \Diamond \bigwedge_{i=1,\ldots,m} \varphi_i$. Therefore, $\mathcal{M}_1, w_1 \models \Diamond \bigwedge_{i=1,\ldots,m} \varphi_i$. But this is a contradiction, since the only states accessible from w_1 are $w^1, \ldots, w^m$, and for each w^i there is a φ_i such that $\mathcal{M}_1, w^i \not\models \varphi_i$. The proof of the zag condition is similar. QED

The modal invariance Lemma (Lemma 17) can be used to prove what can and cannot be expressed in the basic modal language.

Fact 19 *Let $\mathcal{M} = \langle W, R, V\rangle$ be a relational model. The* universal operator *is a unary operator $[A]\varphi$ defined as follows:*

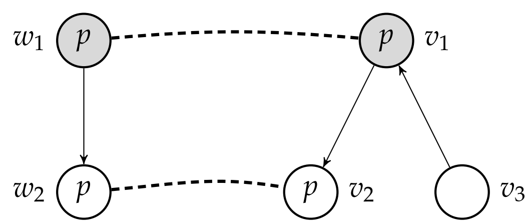
$$\mathcal{M}, w \models [A]\varphi \textit{ iff for all } v \in W, \mathcal{M}, v \models \varphi$$

The universal operator $[A]$ is not definable in the basic modal language.

Proof. Suppose that the universal operator is definable in the basic modal language. Then there is a basic modal formula $\alpha(\cdot)$ such[4] that for any formula φ and any relational

[4]The notation $\alpha(\cdot)$ means that α is a basic modal formula with "free slots" such that $\alpha(\varphi)$ is a well formed modal formula with φ plugged into the free slots.

structure $\mathcal{M}$ with state w, we have $\mathcal{M}, w \models [A]\varphi$ iff $\mathcal{M}, w \models \alpha(\varphi)$. Consider the relational model $\mathcal{M} = \langle W, R, V\rangle$ with $W = \{w_1, w_2\}$, $R = \{(w_1, w_2)\}$ and $V(w_1, p) = V(w_2, p) = T$. Note that $\mathcal{M}, w_1 \models \models [A]p$. Since the universal operator is assumed to be defined by $\alpha(\cdot)$, we must have $\mathcal{M}, w_1 \models \alpha(p)$. Consider the relational model $\mathcal{M}' = \langle W', R', V'\rangle$ with $W' = \{v_1, v_2, v_3\}$, $R' = \{(v_1, v_2), (v_3, v_1)\}$ and $V'(v_1, p) = V'(v_2, p) = T$. Note that $Z = \{(w_1, v_2), (w_2, v_2)\}$ is a bismulation relating w_1 and v_1 (i.e., $\mathcal{M}, w_1 \underline{\leftrightarrow} \mathcal{M}', v_1$). These relational models and bisimulation is pictured below:



By Lemma 17, $\mathcal{M}, w_1 \leftrightsquigarrow \mathcal{M}', v_1$. Therefore, since $\alpha(p)$ is a formula of the basic modal language and $\mathcal{M}, w_1 \models \alpha(p)$, we have $\mathcal{M}', v_1 \models \alpha(p)$. Since $\alpha(p)$ defines the universal operator, $\mathcal{M}', v_1 \models [A]p$, which is a contradiction. Hence, $[A]$ is not definable in the basic modal language. QED

Fact 20 *Let $\mathcal{M} = \langle W, R, V\rangle$ be a relational model. Define the "exists two" operator $\Diamond_2\varphi$ as follows:*

$$\mathcal{M}, w \models \Diamond_2\varphi \text{ iff there is } v_1, v_2 \in W \text{ such that } v_1 \neq v_2,\ \mathcal{M}, v_1 \models \varphi \text{ and } \mathcal{M}, v_2 \models \varphi$$

The exist two $\Diamond_2$ operator is not definable in the basic modal language.

Proof. Suppose that the $\Diamond_2$ is definable in the basic modal language. Then there is a basic modal formula $\alpha(\cdot)$ such that for any formula φ and any relational model $\mathcal{M}$ with state w, we have $\mathcal{M}, w \models \Diamond_2\varphi$ iff $\mathcal{M}, w \models \alpha(\varphi)$. Consider the relational model $\mathcal{M} = \langle W, R, V\rangle$ with $W = \{w_1, w_2, w_3\}$, $R = \{(w_1, w_2), (w_1, w_3)\}$ and $V(p) = \{w_2, w_3\}$. Note that $\mathcal{M}, w_1 \models \Diamond_2 p$. Since $\Diamond_2$ is assumed to be defined by $\alpha(\cdot)$, we must have $\mathcal{M}, w_1 \models \alpha(p)$. Consider the relational model $\mathcal{M}' = \langle W', R', V'\rangle$ with $W' = \{v_1, v_2\}$, $R' = \{(v_1, v_2)\}$ and $V'(p) = \{v_2\}$. Note that $Z = \{(w_1, v_1), (w_2, v_2), (w_3, v_2)\}$ is a bismulation relating w_1 and v_1 (i.e., $\mathcal{M}, w_1 \underline{\leftrightarrow} \mathcal{M}', v_1$). By Lemma 17, $\mathcal{M}, w_1 \leftrightsquigarrow \mathcal{M}', v_1$. Therefore, since $\alpha(p)$ is a formula of the basic modal language and $\mathcal{M}, w_1 \models \alpha(p)$, we have $\mathcal{M}', v_1 \models \alpha(p)$. Since $\alpha(\cdot)$ defines $\Diamond_2$, $\mathcal{M}', v_1 \models \Diamond_2 p$, which is a contradiction. Hence, $\Diamond_2$ is not definable in the basic modal language. QED

3.1 Defining Classes of Structures

The basic modal language can also be used to define *classes* of structures.

Suppose that P is a property of relations (eg., reflexivity or transitivity). We say a frame $\mathcal{F} = \langle W, R\rangle$ has property P provided R has property P. For example,

- $\mathcal{F} = \langle W, R \rangle$ is called a **reflexive frame** provided R is reflexive, i.e., for all $w \in W$, wRw.
- $\mathcal{F} = \langle W, R \rangle$ is called a **transitive frame** provided R is transitive, i.e., for all $w, x, v \in W$, if wRx and xRv then wRv.

Definition 21 (Defining a Class of Frames) A modal formula φ **defines the class of frames with property** P provided for all frames $\mathcal{F}$, $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ has property P. ◁

Remark 22 (Remark on validity on frames) Note that if $\mathcal{F} \models \varphi$ where φ is some modal formula, then $\mathcal{F} \models \varphi^*$ where φ^* is any **substitution instance** of φ. That is, φ^* is obtained by replacing sentence letters in φ with modal formulas. In particular, this means, for example, that in order to show that $\mathcal{F} \not\models \Box\varphi \rightarrow \varphi$ it is enough to show that $\mathcal{F} \not\models \Box p \rightarrow p$ where p is a sentence letter. (This will be used in the proofs below).

Fact 23 $\Box\varphi \rightarrow \varphi$ *defines the class of reflexive frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Box\varphi \rightarrow \varphi$ iff $\mathcal{F}$ is reflexive.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R \rangle$ is reflexive and let $\mathcal{M} = \langle W, R, V \rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Box\varphi \rightarrow \varphi$. Suppose that $\mathcal{M}, w \models \Box\varphi$. Then for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$. Since R is reflexive, we have wRw. Hence, $\mathcal{M}, w \models \varphi$. Therefore, $\mathcal{M}, w \models \Box\varphi \rightarrow \varphi$, as desired.

($\Rightarrow$) We argue by contraposition. Suppose that $\mathcal{F}$ is not reflexive. We must show $\mathcal{F} \not\models \Box\varphi \rightarrow \varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Box p \rightarrow p$ for some sentence letter p. Since $\mathcal{F}$ is not reflexive, there is a state $w \in W$ such that it is not the case that wRw. Consider the model $\mathcal{M} = \langle W, R, V \rangle$ based on $\mathcal{F}$ with $V(p) = \{v \mid v \neq w\}$. Then $\mathcal{M}, w \models \Box p$ since, by assumption, for all $v \in W$ if wRv, then $v \neq w$ and so $v \in V(p)$. Also, notice that by the definition of V, $\mathcal{M}, w \not\models p$. Therefore, $\mathcal{M}, w \models \Box p \wedge \neg p$, and so, $\mathcal{F} \not\models \Box p \rightarrow p$.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Box\varphi \rightarrow \varphi$. We must show that for all x if xRx. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid xRu\}$. Since $\Box p$ is true at x we also have p true at x. This means that $x \in V(p)$, hence, xRx. QED

Fact 24 $\Box\varphi \rightarrow \Box\Box\varphi$ *defines the class of transitive frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Box\varphi \rightarrow \Box\Box\varphi$ iff $\mathcal{F}$ is transitive.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R \rangle$ is transitive and let $\mathcal{M} = \langle W, R, V \rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Box\varphi \rightarrow \Box\Box\varphi$. Suppose that $\mathcal{M}, w \models \Box\varphi$. We must show $\mathcal{M}, w \models \Box\Box\varphi$. Suppose that $v \in W$ and wRv. We must show $\mathcal{M}, v \models \Box\varphi$. To that end, let $x \in W$ be any state with vRx. Since R is transitive and wRv and vRx, we have wRx. Since $\mathcal{M}, w \models \Box\varphi$, we have $\mathcal{M}, x \models \varphi$. Therefore, since x is an arbitrary state accessible from v, $\mathcal{M}, v \models \Box\varphi$. Hence, $\mathcal{M}, w \models \Box\Box\varphi$, and so, $\mathcal{M}, w \models \Box\varphi \to \Box\Box\varphi$, as desired.

($\Rightarrow$, *by contraposition*) We argue by contraposition. Suppose that $\mathcal{F}$ is not transitive. We must show $\mathcal{F} \not\models \Box\varphi \to \Box\Box\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Box p \to \Box\Box p$ for some sentence letter p. Since $\mathcal{F}$ is not transitive, there are states $w, v, x \in W$ with wRv and vRx but it is not the case that wRx. Consider the model $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$ with $V(p) = \{y \mid y \neq x\}$. Since $\mathcal{M}, x \not\models p$ and wRv and vRx, we have $\mathcal{M}, w \not\models \Box\Box p$. Furthermore, $\mathcal{M}, w \models \Box p$ since the only state where p is false is x and it is assumed that it is not the case that wRx. Therefore, $\mathcal{M}, w \models \Box p \wedge \neg\Box\Box p$, and so, $\mathcal{F} \not\models \Box p \to \Box\Box p$, as desired.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Box\varphi \to \Box\Box\varphi$. We must show that for all x, y, z if xRy and yRz then xRz. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid xRu\}$. Since $\Box p$ is true at x we also have $\Box\Box p$ true at x. This means that for all y if xRy then (for all z if yRz we have $z \in V(p)$). Recall that $z \in V(p)$ means that xRz. Putting everything together we have: for all y if xRy then for all z if yRz then xRz. QED

Fact 25 $\Diamond\Box\varphi \to \Box\Diamond\varphi$ *defines the confluence property: for all x, y, z if xRy and xRz then there is a s such that yRs and zRs.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Diamond\Box\varphi \to \Box\Diamond\varphi$ iff $\mathcal{F}$ satisfies the confluence property: for all x, y, z if xRy and xRz then there is a s such that yRs and zRs.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ satisfies confluence and let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Diamond\Box\varphi \to \Box\Diamond\varphi$. Suppose that $\mathcal{M}, w \models \Diamond\Box\varphi$. We must show $\mathcal{M}, w \models \Box\Diamond\varphi$. Suppose that $x \in W$ with wRx. Since $\mathcal{M}, w \models \Diamond\Box\varphi$, there is a y such that wRy and $\mathcal{M}, y \models \Box\varphi$. Since wRx and wRy, by the confluence property, there is a $s \in W$ with xRs and yRs. Since yRs and $\mathcal{M}, y \models \Diamond\varphi$, we have $\mathcal{M}, s \models \varphi$. Then, since xRs, we have $\mathcal{M}, x \models \Diamond\varphi$. Hence, $\mathcal{M}, w \models \Box\Diamond\varphi$, as desired.

($\Rightarrow$, *by contraposition*) We argue by contraposition. Suppose that $\mathcal{F}$ does not satisfy confluence. We must show $\mathcal{F} \not\models \Diamond\Box\varphi \to \Box\Diamond\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Diamond\Box p \to \Box\Diamond p$ for some sentence letter p. Since $\mathcal{F}$ does not satisfy confluence, there are states $w, x, y \in W$ with wRx and wRy but there is no s such that xRs and yRs. Consider the model $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$ with $V(p) = \{v \mid yRv\}$. Then, $\mathcal{M}, y \models \Box p$ (since all states accessible from y satisfy p). Since there is no s such that xRs and yRs, we also have $\mathcal{M}, x \not\models \Diamond p$. Since wRx and wRy, we have $\mathcal{M}, w \not\models \Box\Diamond p$ and $\mathcal{M}, w \models \Diamond\Box p$. Hence, $\Diamond\Box p \to \Box\Diamond p$ is not valid.

($\Rightarrow$, *directly*) Suppose that $\mathcal{F} \models \Diamond\Box\varphi \to \Box\Diamond\varphi$. We must show that for all x, y, z if xRy and xRz, then there is a s such that yRs and zRs. Let x be any state and consider a model $\mathcal{M}$ based on $\mathcal{F}$ with a valuation $V(p) = \{u \mid yRu\}$. Let y, z be states with xRy and xRz. Since,

$\mathcal{M}, y \models \Box p$, we have $\mathcal{M}, x \models \Diamond\Box p$. This means that $\mathcal{M}, x \models \Box\Diamond p$. Hence, since xRz, we have $\mathcal{M}, z \models \Diamond p$. Thus, there is a states v such that zRv and $v \in V(p)$. Since $v \in V(p)$, we have yRv. Putting everything together we have: for all x, y, z if xRy and xRz, then there is a s such that yRs and zRs. QED

Exercise 6 *Determine which class of frames are defined by the following modal formulas.*

1. $\Box\varphi \to \Diamond\varphi$
2. $\Diamond\varphi \to \Box\varphi$
3. $\varphi \to \Box\Diamond\varphi$
4. $\neg\Box\varphi \to \Box\neg\Box\varphi$
5. $\Box(\Box\varphi \to \varphi)$
6. $\Box\Box\varphi \to \Box\varphi$

4 Normal Modal Logics

Recall the definition of a substitution for modal formulas (Definition 3).

Definition 26 (Tautology) A modal formula φ is called a **(propositional) tautology** if $\varphi = (\alpha)^\sigma$ where σ is a substition, α is a formula of propositional logic and α is a tautology. ◁

For example, $\Box p \to (\Diamond(p \wedge q) \to \Box p)$ is a tautology because $a \to (b \to a)$ is a tautology in the language of propositional logic and

$$(a \to (b \to a))^\sigma = \Box p \to (\Diamond(p \wedge q) \to \Box p)$$

where $\sigma(a) = \Box p$ and $\sigma(b) = \Diamond(p \wedge q)$.

The definition of the minimal normal modal logic **K** is given in Section **??**. The following axiom schemes have played an important role in both the mathematical development of modal logic and in applications of modal logic.

(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	(4)	$\Box\varphi \to \Box\Box\varphi$
(D)	$\Box\varphi \to \Diamond\varphi$	(5)	$\neg\Box\varphi \to \Box\neg\Box\varphi$
(T)	$\Box\varphi \to \varphi$	(L)	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$

Each of the above formulas are called **axiom schemas** and I will often refer to **instances** of these axiom schemas. The general idea is to treat the 'φ' in the above formulas as a meta-variable that can be replaced by specific formulas from $\mathcal{L}$. For instance, $\Box\Diamond p \to \Diamond p$ is a substitution instance of the axiom scheme (T).

The minimal normal modal logic, **K**, is the smallest set of formulas that contains all tautologies, all instances of (K), all instances of (Dual), and is closed under the rules (Nec) (from φ infer $\Box\varphi$) and Modus Ponens (from φ and $\varphi \to \psi$ infer ψ). Other normal modal logics are defined by adding all instances of axiom schema or rules to **K**. If $A_1, \ldots, A_n$ are axiom schemas, then $\mathbf{K} + A_1 + A_2 + \cdots + A_n$ is the smallest set of formulas that contains all tautologies, all instances of K, all instances of *Dual*,[5] for each $i = 1, \ldots, n$, all instances of A_i, and is closed under the rules (Nec) (from φ infer $\Box\varphi$) and (MP) (from φ and $\varphi \to \psi$ infer ψ).

Remark 27 (Rules) Of course, one may also be interested in defining modal logics by adding new rules to **K**. Similar notation can be used to define extensions of **K** with new rules—e.g., if R is a rule then $\mathbf{K} + R$ is the smallest set of formulas that contain **K** and is closed under the rule R.

Using the above naming convention for logics, I can now define a number of well-studied normal modal logics:

$$\begin{array}{rcl} \mathbf{T} & \text{is} & \mathbf{K} + (\mathsf{T}) \\ \mathbf{S4} & \text{is} & \mathbf{K} + (\mathsf{T}) + (4) \\ \mathbf{S5} & \text{is} & \mathbf{K} + (\mathsf{T}) + (4) + (5) \\ \mathbf{KD45} & \text{is} & \mathbf{K} + (\mathsf{D}) + (4) + (5) \\ \mathbf{GL} & \text{is} & \mathbf{K} + (\mathsf{L}) \end{array}$$

Definition 28 (Deduction) Suppose that **L** is an extension of **K** defined from axiom schemas $A_1, \ldots, A_k$. A **deduction** in **L** is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. α_i is a tautology
2. α_i is an instance of K
3. α_i is an instance of A_j for some $j = 1, \ldots, k$
4. α_i is of the form $\Box\alpha_j$ for some $j < i$
5. α_i follows by Modus Ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$).

Write $\vdash_{\mathbf{K}} \varphi$ if there is a deduction containing φ (i.e., in which φ is the last formula in a deduction). ◁

[5] The axiom schema (Dual), i.e., $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$, is needed when $\Box$ and $\Diamond$ are treating as basic operators in the language (rather than taking one to be a defined operator).

Fact 29 $\vdash_{\mathbf{K}} (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

Proof.

1.	$\varphi \to (\psi \to (\varphi \wedge \psi))$	propositional tautology
2.	$\Box(\varphi \to (\psi \to (\varphi \wedge \psi)))$	(Nec) 1
3.	$\Box(\varphi \to (\psi \to (\varphi \wedge \psi))) \to (\Box\varphi \to \Box(\psi \to (\varphi \wedge \psi)))$	instance of (K)
4.	$\Box\varphi \to \Box(\psi \to (\varphi \wedge \psi))$	(MP) 2,3
5.	$\Box(\psi \to (\varphi \wedge \psi)) \to (\Box\psi \to \Box(\varphi \wedge \psi)$	instance of (K)
6.	$(a \to b) \to ((b \to c) \to (a \to c))$	propositional tautology $a := \Box\varphi, b := \Box(\psi \to (\varphi \wedge \psi)),$ $c := \Box\psi \to \Box(\varphi \wedge \psi)$
7.	$(b \to c) \to (a \to c)$	(MP) 4,6 $a := \Box\varphi, b := \Box(\psi \to (\varphi \wedge \psi)),$ $c := \Box\psi \to \Box(\varphi \wedge \psi)$
8.	$\Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi))$	(MP) 5,7
9.	$(a \to (b \to c)) \to ((a \wedge b) \to c)$	propositional tautology $a := \Box\varphi, b := \Box\psi, c = \Box(\varphi \wedge \psi),$
10.	$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$	(MP) 8, 9

QED

Fact 30 $\vdash_{\mathbf{K}} \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$

Proof.

1.	$\varphi \wedge \psi \to \varphi$	propositional tautology
2.	$\Box((\varphi \wedge \psi) \to \varphi)$	(Nec) 1
3.	$\Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\varphi)$	instance of (K)
4.	$\Box(\varphi \wedge \psi) \to \Box\varphi$	(MP) 2,3
5.	$\varphi \wedge \psi \to \psi$	propositional tautology
6.	$\Box((\varphi \wedge \psi) \to \psi)$	(Nec) 5
7.	$\Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\psi)$	instance of (K)
8.	$\Box(\varphi \wedge \psi) \to \Box\psi$	(MP) 5,6
9.	$(a \to b) \to ((a \to c) \to (a \to (b \wedge c)))$	propositional tautology $(a := \Box(\varphi \wedge \psi), b := \Box\varphi, c := \Box\psi)$
10.	$(a \to c) \to (a \to (b \wedge c))$	(MP) 4,9
11.	$\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$	(MP) 8,10

QED

Definition 31 (Deduction with Assumptions) Suppose that Γ is a set of modal formulas and $\mathbf{L}$ is an extension of $\mathbf{K}$. We say that φ is deducible from Γ provided that there are finitely many formulas $\alpha_1, \ldots, \alpha_k \in \Gamma$ such that $\vdash_{\mathbf{L}} (\alpha_1 \wedge \cdots \wedge \alpha_k) \to \varphi$. ◁

Remark 32 (Comments on Necessitation) Note that the side condition in item 4. in the above definition is crucial. Without it, one application of Necessitation shows that $\{p\} \vdash_{\mathbf{K}} \Box p$. Using a deduction theorem stating that $\Sigma; \alpha \vdash_{\mathbf{K}} \beta$ implies $\Sigma \vdash_{\mathbf{K}} \alpha \to \beta$, we can conclude that $\vdash_{\mathbf{K}} p \to \Box p$. But, clearly $p \to \Box p$ cannot be a theorem of **K** (why?).

Definition 33 (Semantic Consequence) Suppose that Γ is a set of modal formulas and F is a class of relational frames. We say φ is a **semantic consequence** of Γ with respect to F, denoted $\Gamma \models_{\mathsf{F}} \varphi$, provided for all models $\mathcal{M} = \langle W, R, V\rangle$ based on a frame from F (i.e., $\langle W, R\rangle \in \mathsf{F}$) and all states $w \in W$, if $\mathcal{M}, w \models \Gamma$, then $\mathcal{M}, w \models \varphi$ (where $\mathcal{M}, w \models \Gamma$ when $\mathcal{M}, w \models \gamma$ for all $\gamma \in \Gamma$). ◁

Definition 34 (Soundness, Weak/Strong Completeness) Suppose that F is a class of relational frames. A logic **L** is **sound** with respect to F provided, for all sets of formulas Γ, if $\Gamma \vdash_{\mathbf{L}} \varphi$, then $\Gamma \models_{\mathsf{F}} \varphi$. A logic **L** is **strongly complete** with respect to F provided for all sets of formulas Γ, if $\Gamma \models_{\mathsf{F}} \varphi$, then $\Gamma \vdash_{\mathbf{L}} \varphi$. Finally, a logic **L** is **weakly complete** with respect to F provided that for all $\varphi \in \mathcal{L}$, if $\models_{\mathsf{F}} \varphi$, then $\vdash_{\mathbf{L}} \varphi$. ◁

Clearly, if a logic is strongly complete then it is weakly complete. Interestingly, the converse is not true (as we will see below). The proofs of the following theorem can be found in Blackburn et al. (2001).

Theorem 35 (Completeness Theorems)

- **K** *is sound and strongly complete with respect to the class of all relational frames.*
- **T** *is sound and strongly complete with respect to the class of reflexive relational frames.*
- **S4** *is sound and strongly complete with respect to the class of reflexive and transitive relational frames.*
- **S5** *is sound and strongly complete with respect to the class of reflexive, transitive and Euclidean relational frames (i.e., relations that form a partition).*
- **KD45** *is sound and strongly complete with respect to the class of serial, transitive and Euclidean relational frames (i.e., relations that form a* quasi-partition*).*

The logic **GL** does not follow the same pattern as the logics mentioned in the above theorem. There is a natural class of relational frames that characterizes **GL**. A relation $R \subseteq W \times W$ is **converse well-founded** (also called Noetherian) if there is no infinite ascending chain of states—i.e., there is no infinite set of distinct elements $w_0, w_1, \ldots$ from W, such that $w_0 \; R \; w_1 \; R \; w_2 \cdots$. Note that if R is converse well-founded, then it is **irreflexive** (for all $w \in W$, $w \not R w$). It is not hard to see that **G** is sound with respect to the class of frames that are transitive and converse well-founded. However, **GL** is *not* strongly complete with respect to this class of frames. To see this, we need some additional notation.

Definition 36 (Compactness) Suppose that **L** is sound with respect to some class of frames F. We say that **L** is **compact** provided that for any set of formulas Γ, if Γ is finitely satisfiable (every finite subset of formulas is satisfiable), then Γ is satisfiable. $\triangleleft$

Proposition 37 *If* **L** *is sound and strongly complete with respect to some class of frames* F*, then* **L** *is compact.*

Proof. Suppose that **L** is sound and strongly complete with respect to some class of frames F. Suppose that Γ is any set of formulas that is finitely satisfiable. I.e., every finite subset $\Gamma_0 \subseteq \Gamma$ has a model (based on a frame from F). If Γ is not satisfiable, then, since every consistent set is satisfiable, Γ is inconsistent. I.e., $\Gamma \vdash_{\mathbf{L}} \bot$. This means that there is a deduction from Γ in **L** of $\bot$. Since deductions are finite in length, only finitely many assumptions from Γ can be used in the deduction. This means that there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathbf{L}} \bot$. By soundness, this means that Γ_0 is not satisfiable. This contradicts our assumption. Thus Γ is satisfiable. QED

Observation 38 *The logic* **GL** *is not strongly complete with respect to the class of transitive and converse well-founded relational frames.*

Proof. We will show that **GL** is not compact. Then, by Proposition 37, we can conclude that **GL** is not strongly complete. Suppose that

$$\Gamma = \{\Diamond p_0, \Box(p_0 \to \Diamond p_1), \Box(p_1 \to \Diamond p_2), \ldots, \Box(p_n \to \Diamond p_{n+1}), \ldots\}.$$

Suppose that $\Gamma_0 \subseteq \Gamma$ is finite. We will show that Γ_0 is satisfiable. First of all, note that without loss of generality we can assume that $\Gamma_0 = \{\Diamond p_0, \Box(p_0 \to \Diamond p_1), \Box(p_1 \to \Diamond p_2), \ldots, \Box(p_{k-1} \to \Diamond p_k)\}$. (If $\Diamond p_0 \notin \Gamma_0$, then since Γ_0 only contains formulas with $\Box$ as the main connective, and so, a single state with no accessible worlds will make all the formulas in Γ_0 true.) We can construct a model $\mathcal{M} = \langle W, R, V\rangle$ with a state that makes all of Γ_0 true. Suppose that $W = \{w, w_0, w_1, \ldots, w_k\}$ and let R be the transitive closure of

$$w\ R\ w_0\ R\ w_1 \cdots w_{k-1}\ R\ w_k$$

That is, R is the smallest transitive relation that contains

$$R_0 = \{(w, w_0), (w_0, w_1), \ldots, (w_j, w_{j+1}), \ldots, (w_{k-1}, w_k)\}.$$

Furthermore, suppose that $V : \{p_0, \ldots, p_k\} \to \wp(W)$ is the valuation function defined as follows: $V(p_i) = \{w_i\}$ for $i = 0, \ldots, k$. Then, since $\mathcal{M}, w_0 \models p_0$ and $w\ R\ w_0$, we have $\mathcal{M}, w \models \Diamond p_0$. Furthermore, if $w' \in W$ is a state such that $w\ R\ w'$ then $w' = w_i$ for some $i = 0, \ldots, k$. If $i \neq 0$, then $\mathcal{M}, w' \not\models p_0$. Thus, trivially, $\mathcal{M}, w' \models p_0 \to \Diamond p_1$. If $i = 0$, then, since $w_0\ R\ w_1$ and $\mathcal{M}, w_1 \models p_1$, we have that $\mathcal{M}, w_0 \models p_0 \to \Diamond p_1$. Thus, $\mathcal{M}, w_0 \models p_0 \to \Diamond p_1$. Hence, $\mathcal{M}, w \models \Box(p_0 \to \Diamond p_1)$. A similar argument shows that $\mathcal{M}, w \models \Box(p_j \to \Diamond p_{j+1})$ for $j = 0, \ldots, k-1$. Thus, $\mathcal{M}, w$ satisfies Γ_0.

However, it is not hard to see that there is no that is transitive and converse well-founded model with a state satisfying all of Γ. Suppose that there is a model $\mathcal{M} = \langle W, R, V \rangle$ and state $w \in W$ such that $\mathcal{M}, w \models \varphi$ for all $\varphi \in \Gamma$. Since $w \models \Diamond p_0$ there must be some accessible world w' such that $\mathcal{M}, w' \models p_0$. It must be the case that $w' \neq w$ (otherwise, R is not converse well-founded). Since $\mathcal{M}, w \models \Box(p_0 \to \Diamond p_1)$ and $w \; R \; w'$, we must have $\mathcal{M}, w' \models p_0 \to \Diamond p_1$. Hence there is some world w'' such that $w' \; R \; w''$ and $\mathcal{M}, w'' \models p_1$. Since R is transitive, we must have $w \; R \; w''$. Since R is converse well-founded, we must have $w'' \neq w$. Continuing in this manner, we construct an infinite chain of worlds that are R-accessible, contradicting the assumption that R is converse well-founded. Thus, Γ is not satisfiable on any model that is converse well-founded. QED

Nonetheless, Segerberg (1971) proved a *weak* completeness theorem for **GL**. The proof is beyond the scope of this Appendix (see Blackburn et al. (2001) for the details).

Theorem 39 *The logic* **GL** *is sound and weakly complete with respect to the class of transitive and converse well-founded frames.*

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