PHIL309P
Philosophy, Politics and Economics

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Announcements

- Course website
  https://myelms.umd.edu/courses/1133211
- Reading
  - Gaus, Ch. 5
  - EP, Voting Methods (Stanford Encyclopedia of Philosophy)
  - C. List, Social Choice Theory (Stanford Encyclopedia of Philosophy)
  - M. Morreau, Arrow's Theorem (Stanford Encyclopedia of Philosophy)
- Online videos
- Problem set 3

Arrow's Theorem

K. Arrow. *Social Choice and Individual Values*. John Wiley & Sons, 1951.

Arrow's Theorem

Let X be a finite set with *at least three elements* and N a finite set of n voters.

Social Welfare Function: $F : \mathcal{D} \rightarrow O(X)$ where $\mathcal{D} \subseteq O(X)^n$

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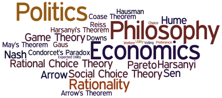

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Reminders:

- $O(X)$ is the set of transitive and complete relations on X
- For $R \in O(X)$, let P_R denote the strict subrelation and I_R the indifference subrelation:
  - $A \; P_R \; B$ iff $A \; R \; B$ and not $B \; R \; A$
  - $A \; I_R \; B$ iff $A \; R \; B$ and $B \; R \; A$

Unanimity

$F : \mathcal{D} \to O(X)$

If each agent ranks A above B, then so does the social ranking.

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For all profiles $\mathbf{R} = (R_1, \ldots, R_n) \in \mathcal{D}$:

$$\text{If for each } i \in N,\ A\ R_i^{>}\ B \text{ then } A\ P_{F(\mathbf{R})^{>}}\ B$$

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Voter's are free to choose any preference they want.

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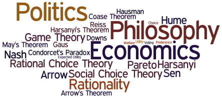
The domain of F is the set of *all* profiles, i.e., $\mathcal{D} = O(X)^n$.

Independence of Irrelevant Alternatives

$F : \mathcal{D} \rightarrow O(X)$

The social ranking (higher, lower, or indifferent) of two alternatives *A* and *B* depends only the relative rankings of *A* and *B* for each voter.

Independence of Irrelevant Alternatives



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For all profiles $\mathbf{R} = (R_1, \ldots, R_n)$ and $\mathbf{R}' = (R'_1, \ldots, R'_n)$:

If $R_{i\{A,B\}} = R'_{i\{A,B\}}$ for all $i \in N$, then $F(\mathbf{R})_{\{A,B\}}$ iff $F(\mathbf{R}')_{\{A,B\}}$.

where $R_{\{X,Y\}} = R \cap \{X, Y\} \times \{X, Y\}$

IIA For all profiles $\mathbf{R} = (R_1, \ldots, R_n)$ and $\mathbf{R}' = (R'_1, \ldots, R'_n)$:

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Dictatorship

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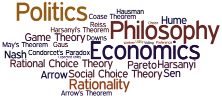
There is a $d \in N$ such that for each profile $\mathbf{R} = (R_1, \ldots, R_d, \ldots, R_n)$, if $A\ R_d^{\succ}\ B$, then $A\ P_{F(\mathbf{R})^{\succ}}\ B$

M. Morreau. *Arrow's Theorem*. Stanford Encyclopedia of Philosophy, 2014.

Arrow’s Theorem



Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Arrow's Theorem

D. Campbell and J. Kelly. *Impossibility Theorems in the Arrovian Framework*. Handbook of Social Choice and Welfare Volume 1, pgs. 35 - 94, 2002.

W. Gaertner. *A Primer in Social Choice Theory*. Oxford University Press, 2006.

J. Geanakoplos. *Three Brief Proofs of Arrow's Impossibility Theorem*. Economic Theory, **26**, 2005.

P. Suppes. *The pre-history of Kenneth Arrow's social choice and individual values*. Social Choice and Welfare, 25, pgs. 319 - 326, 2005.

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Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\mathbf{R}|_S$ denote the restriction of the profile to candidates in S.

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Binary Independence: For all profiles $\mathbf{R}, \mathbf{R}'$ and candidates $A, B \in X$:

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***m*-Ary Independence**: For all profiles $\mathbf{R}, \mathbf{R}'$ and for all $S \subseteq X$ with $|S| = m$:

$$\text{If } \mathbf{R}|_S = \mathbf{R}'|_S, \text{ then } F(\mathbf{R})|_S = F(\mathbf{R}')|_S$$

Weakening IIA

Theorem. (Blau) Suppose that $m = 2, \ldots, |X| - 1$. If a social welfare function F satisfies m-ary independence, then it also satisfies binary independence.

J. Blau. *Arrow's theorem with weak independence*. Economica, 38, pgs. 413 - 420, 1971.

S. Cato. *Independence of Irrelevant Alternatives Revisited*. Theory and Decision, 2013.

Arrow's Theorem

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Weakening Unanimity

$F : \mathcal{D} \to O(X)$

Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A\ P_d\ B$, then $A\ P_{F(\mathbf{R})}\ B$

Inversely Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A\ P_d\ B$, then $B\ P_{F(\mathbf{R})}\ A$

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Null: For all $A, B \in X$ and for all $\mathbf{R} \in \mathcal{D}$: $A \; I_{F(\mathbf{R})} \; B$

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Non-Imposition: For all $A, B \in X$, there is a $\mathbf{R} \in \mathcal{D}$ such that $A\ F(\mathbf{R})\ B$

Weakening Unanimity

Theorem (Wilson) Suppose that N is a finite set. If a social welfare function satisfies universal domain, independence of irrelevant alternatives and non-imposition, then it is either null, dictatorial or inversely dictatorial.

R. Wilson. *Social Choice Theory without the Pareto principle*. Journal of Economic Theory, 5, pgs. 478 - 486, 1972.

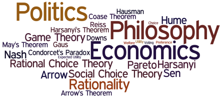
Y. Murakami. *Logic and Social Choice*. Routledge, 1968.

S. Cato. *Social choice without the Pareto principle: A comprehensive analysis*. Social Choice and Welfare, 39, pgs. 869 - 889, 2012.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Social Choice Functions



$F : \mathcal{D} \rightarrow \wp(X) - \emptyset$

Resolute: For all profiles $\mathbf{R} \in \mathcal{D}$, $|F(\mathbf{R})| = 1$

Non-Imposed: For all candidates $A \in X$, there is a $\mathbf{R} \in \mathcal{D}$ such that $F(\mathbf{R}) = \{A\}$.

Monotonicity: For all profiles $\mathbf{R}$ and $\mathbf{R}'$, if $A \in F(\mathbf{R})$ and for all $i \in N$, $\mathbf{N}_{\mathbf{R}}(A\ P_i\ B) \subseteq \mathbf{N}_{\mathbf{R}'}(A\ P'_i\ B)$ for all $B \in X - \{A\}$, then $A \in F(\mathbf{R}')$.

Dictator: A voter d is a dictator if for all $\mathbf{R} \in \mathcal{D}$, $F(\mathbf{R}) = \{A\}$, where A is d's top choice.

Social Choice Functions

Muller-Satterthwaite Theorem. Suppose that there are more than three alternatives and finitely many voters. Every resolute social choice function $F : L(X)^n \to X$ that is monotonic and non-imposed is a dictatorship.

E. Muller and M.A. Satterthwaite. *The Equivalence of Strong Positive Association and Strategy-Proofness*. Journal of Economic Theory, 14(2), pgs. 412 - 418, 1977.

Arrow’s Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

- Infinitely many voters.
- Domain restrictions.
- Richer ballots.

Universal Domain

Universal Domain: The domain of the social welfare (choice) function is $\mathcal{D} = L(X)^n$ (or $O(X)^n$)

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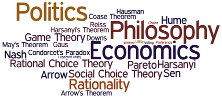

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Epistemic Rationale: "If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings." (Arrow, 1963, pg. 24)

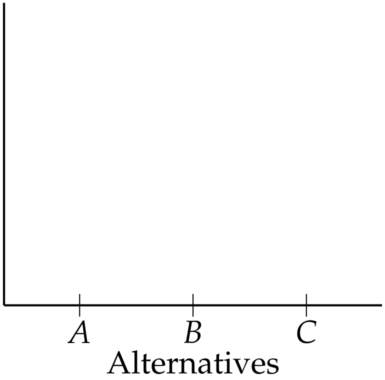
Domain Restrictions

- Single-Peaked preferences
- Sen's Value Restriction
- Assumptions about the distribution of preferences

W. Gaertner. *Domain Conditions in Social Choice Theory*. Cambridge University Press, 2001.

1	1	1
A	B	C
B	C	A
C	A	B

1	1	1
A	B	C
B	C	A
C	A	B



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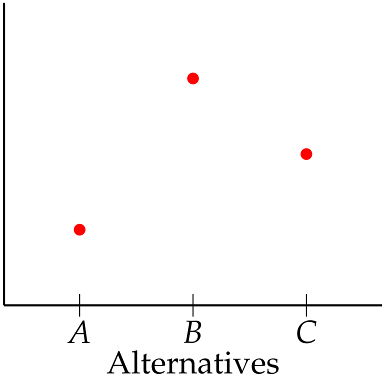

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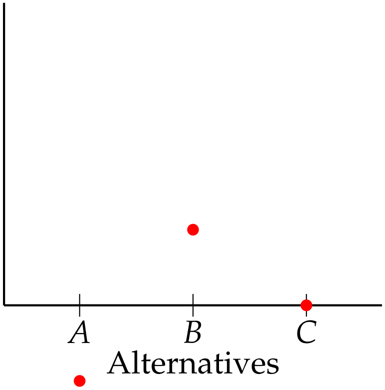

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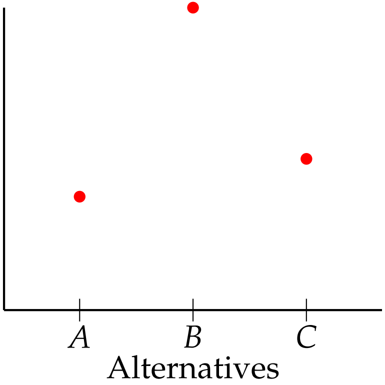

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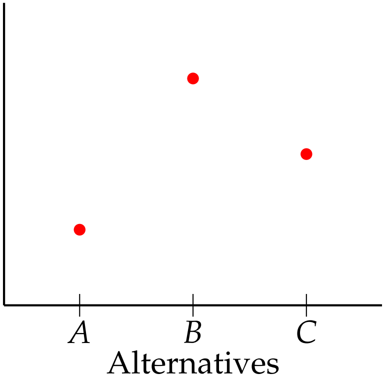
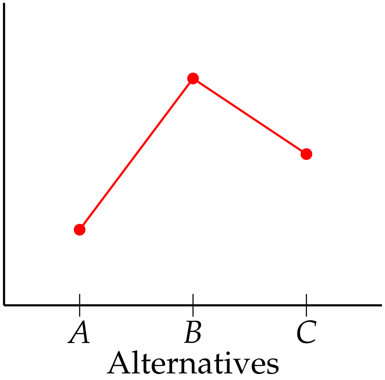


1	1	1
A	$\boxed{B}$	C
B	$\boxed{C}$	A
C	$\boxed{A}$	B



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A	B	C
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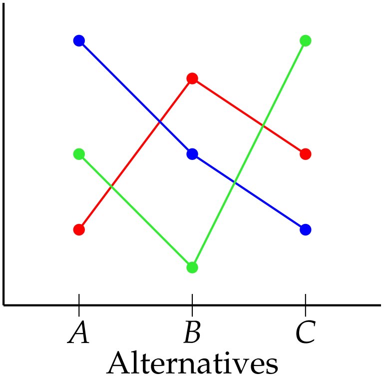
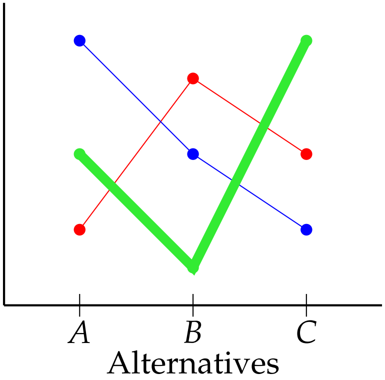


1	1	1
A	B	C
B	C	A
C	A	B



1	1	1
A	B	C
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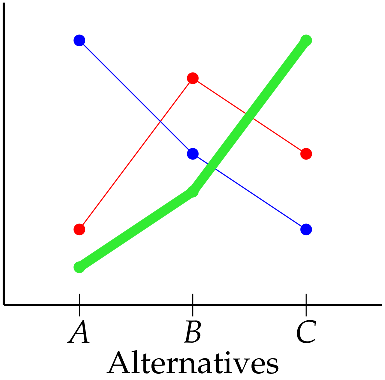
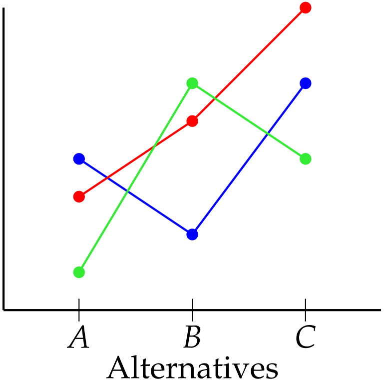
C	A	A



1	1	1
C	C	B
A	B	C
B	A	A



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C	C	B
A	B	C
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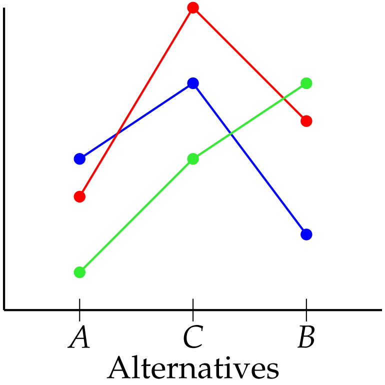

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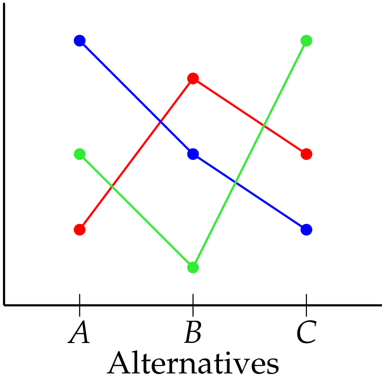

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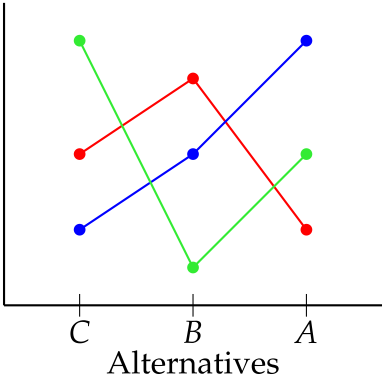

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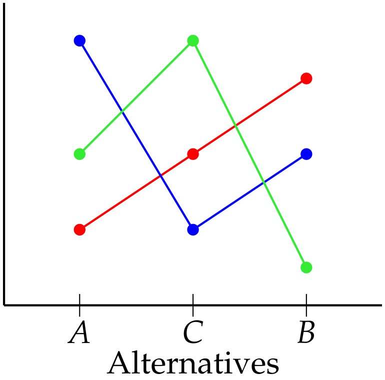

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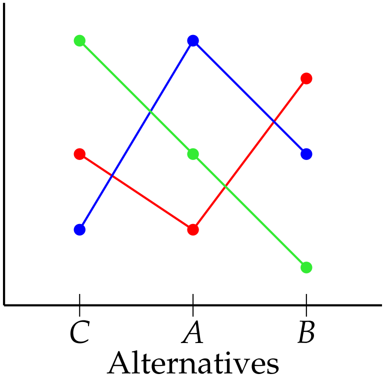
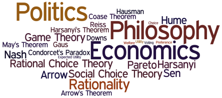


D. Black. *On the rationale of group decision-making*. Journal of Political Economy, 56:1, pgs. 23 - 34, 1948.

Single-Peakedness: the preferences of group members are said to be single-peaked if the alternatives under consideration can be represented as points on a line and each of the utility functions representing preferences over these alternatives has a maximum at some point on the line and slopes away from this maximum on either side.



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Theorem. If there is an odd number of voters that display single-peaked preferences, then a Condorcet winner exists.

D. Miller. *Deliberative Democracy and Social Choice*. Political Studies, 40, pgs. 54 - 67, 1992.

C. List, R. Luskin, J. Fishkin and I. McLean. *Deliberation, Single-Peakedness, and the Possibility of Meaningful Democracy: Evidence from Deliberative Polls*. Journal of Politics, 75(1), pgs. 80 - 95, 2013.

Sen's Value Restriction

A. Sen. *A Possibility Theorem on Majority Decisions*. Econometrica 34, 1966, pgs. 491 - 499.

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Assume n voters (n is odd).

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Triplewise value-restriction: For every triple of distinct candidates A, B, C there exists an $x_i \in \{A, B, C\}$ and $r \in \{1, 2, 3\}$ such that no voter ranks x_i has her rth preference among A, B, C.

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Theorem (Sen, 1966). For every profile satisfying triplewise value-restriction, pairwise majority voting generates a transitive group preference ordering.

Restrict the *distribution* of preferences

M. Regenwetter, B. Grofman, A.A.J. Marley and I. Tsetlin. *Behavioral Social Choice*. Cambridge University Press, 2006.

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