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Quick Completeness Proofs for Some Logics of Conditionals

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Introduction We start from the idea that a conditional $\alpha \rightarrow \beta$ is true iff α & $\sim\beta$ is either an impossibility or at least a *remoter* possibility, in some sense, than α & β. Let us try to make this precise.

First, we fix a *language* for the logic of conditionals: Let $\mathcal{L}$ be the set of formulas obtainable from the variables $p_1, p_2, p_3, \ldots$ using the arrow and the usual truth-functional connectives (viz., the true $\top$, the false $\bot$, negation $\sim$, conjunction &, inclusive disjunction $\vee$, material $\supset$ and $\equiv$). For $A \subseteq \mathcal{L}$ finite, $\bigwedge A$ denotes the conjunction, $\bigvee A$ the disjunction, of all elements of A (suitably grouped); e.g., $\bigwedge\phi = \top$, $\bigvee\phi = \bot$.

Second, we fix a notion of *model*. Let $\mathcal{M}$ be the set of all pairs (W, R), with W a nonempty set and R a trinary relation on it. For $x \in W$, we set $W_x = \{y\colon \exists z\, Rxyz\}$, and we require that R satisfy the following *reflexivity* and *transitivity* requirements:

$$\forall x \in W\ \forall y \in W_x\, Rxyy$$
$$\forall x \in W\ \forall y, z, w \in W_x\, (Rxyz\ \&\ Rxzw \supset Rxyw).$$

A *model-class* is any $\mathcal{N} \subseteq \mathcal{M}$ closed under isomorphism; the interesting examples are obtained by imposing certain *characteristic restrictions* on R.

Next, we fix a notion of *satisfaction/validity*. A *valuation* in $(W, R) \in \mathcal{M}$ is a map V assigning each variable p_i a subset of W. V can be extended to all of $\mathcal{L}$ by treating truth-functions in the usual way (e.g., $V(\sim\alpha) = W - V(\alpha)$, $V(\alpha\ \&\ \beta) = V(\alpha) \cap V(\beta)$), and defining $V(\alpha \rightarrow \beta)$ as the set of all $x \in W$ such that:

$$\forall y \in W_x \cap V(\alpha)\ \exists z \in W_x \cap V(\alpha)\ [Rxzy\ \&\ \forall t \in W_x \cap V(\alpha)(Rxtz \supset t \in V(\beta))]$$

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α is *satisfiable* (resp., *valid*) in (W, R) iff $V(\alpha) \neq \phi$ (resp., $V(\alpha) = W$) for some (resp., all) valuations V. For a model class $\mathcal{N}$, $\lambda(\mathcal{N}) = \{\alpha \in \mathcal{L}: \forall(W, R) \in \mathcal{N}$ (α is valid in (W, R))$\}$.

Intuitively, the W in $(W, R) \in \mathcal{M}$ may be thought of as the set of all possible situations; then $Rxyz$ may be thought of as meaning that from the point of view of x, y is no more remote a possibility than z. A valuation V may be thought of as telling us in which situations a given atomic statement p_i holds; then the rather complicated definition of $V(\alpha \rightarrow \beta)$—which has been taken over from D. K. Lewis—represents one attempt to give precision to the idea 'α & $\sim\beta$ is a remoter possibility than α & β' with which we started. Some recent work of Lewis (I am indebted to him for making it available in advance of publication) serves to show that a number of analyses of conditionals in the literature amount to the endorsement of one or another $\lambda(\mathcal{N})$ as the correct logic of conditionals. We will therefore be interested in the problem of axiomatizing various $\lambda(\mathcal{N})$.

So finally, we fix a basic axiomatic system $\mathcal{J}$. The rules of $\mathcal{J}$ are Substitution, Modus Ponens (MP) for $\supset$, plus the following rule of Replacement of Provable Equivalents (RPE):

RPE Rule From $\gamma \equiv \delta$ and β to infer α, where $\alpha = \psi(\delta)$ differs from $\beta = \psi(\gamma)$ only by replacing some subformulas of β of form γ by δ.

The axioms of $\mathcal{J}$ are all truth-functional tautologies plus:

A0 $p \rightarrow p$
A1 $(p \rightarrow q) \,\&\, (p \rightarrow r) \supset (p \rightarrow q \,\&\, r)$
A2 $(p \rightarrow q \,\&\, r) \supset (p \rightarrow q)$
A3 $(p \rightarrow q) \,\&\, (p \rightarrow r) \supset (p \,\&\, q \rightarrow r)$
A4 $(p \rightarrow r) \,\&\, (q \rightarrow r) \supset (p \vee q \rightarrow r)$.

A *standard-system* is any extension $\mathcal{S}$ of $\mathcal{J}$ obtained by adding finitely many new axioms, called the *characteristic axioms* of $\mathcal{S}$. No tampering with the rules is permitted. Standard systems fall within the scope of some work of Lewis [1]: If the characteristic axioms of such a system $\mathcal{S}$ involve no nesting of arrows, then the set $\theta(\mathcal{S})$ of theses of $\mathcal{S}$ is recursively decidable.

Recently R. Stalnaker (unpublished) has solved the long-open problem of proving $\lambda(\mathcal{M}) = \theta(\mathcal{J})$. His method seems to be special to this one problem, and is devious, taking a detour through an alternative modeling in terms of the 'selection functions' favored by Stalnaker, J. Pollock, and others. The present notes give a straightforward proof that $\lambda(\mathcal{M}) = \theta(\mathcal{J})$ by a method that works directly with Lewis's 'relative remoteness' modeling as introduced above, and is general, in that many (often previously known) results of the type $\lambda(\mathcal{N}) = \theta(\mathcal{S})$ are obtainable by slight modifications (usually simplifications) of the argument. Comments by Lewis and Stalnaker have led to many improvements in the exposition. Very recently F. Veltman (unpublished) announced yet another proof that $\lambda(\mathcal{M}) = \theta(\mathcal{J})$, using yet another kind of modeling, favored by Veltman, A. Kratzer, and others.

We collect here for future reference some theses and derived rules of $\mathcal{J}$. As always, the turnstile ($\vdash$) denotes thesishood:

B0 *If* $\vdash \alpha \supset \beta$, *then* $\vdash \alpha \rightarrow \beta$
B1 *If* $\vdash \beta \supset \gamma$, *then* $\vdash (\alpha \rightarrow \beta) \supset (\alpha \rightarrow \gamma)$
B2 $\vdash (\alpha \rightarrow \beta) \supset (\alpha \vee \gamma \rightarrow \beta \vee \gamma)$
B3 $\vdash (\alpha \rightarrow \gamma) \,\&\, (\alpha \rightarrow \beta) \,\&\, (\beta \rightarrow \alpha) \supset (\beta \rightarrow \gamma)$
B4 *If* $\vdash (\beta \supset \alpha) \,\&\, (\alpha \,\&\, \gamma \supset \beta)$ *then* $\vdash (\alpha \rightarrow \gamma) \supset (\beta \rightarrow \gamma)$.

Readers who wish to work these out for themselves–it is a pleasant exercise–can turn directly to the next section. For the rest, we sketch proofs.

Our work will be easier if we use a form of the Deduction Theorem. Define a *deduction from a set of hypotheses A* in a standard system to be a finite string of formulas, some perhaps bearing stars (*), each satisfying one of the following:

- α is a substitution instance of an axiom
- α is an element of A, and is starred
- α follows by MP from earlier unstarred formulas
- α follows by MP from earlier formulas, and is starred
- α follows by RPE from earlier unstarred formulas
- α follows by RPE from an earlier unstarred formula $\gamma \equiv \delta$ and another earlier formula β, and is starred.

Then any unstarred formula in a deduction from hypotheses is actually a thesis, and it is easily established that:

(DT) *If there exists a deduction of* α *from hypotheses* A, *then* $\vdash \bigwedge A \supset \alpha$.

This established, for B0 we assume $\vdash \alpha \supset \beta$ and consider:

(i)	$\alpha \equiv \beta \,\&\, \alpha$	[$\alpha \supset \beta$]
(ii)	$\alpha \rightarrow \alpha$	[A0]
(iii)	$\alpha \rightarrow \beta \,\&\, \alpha$	[RPE, i, ii]
(iv)	$\alpha \rightarrow \beta$	[A2, iii]

For B1, assume $\vdash \beta \supset \gamma$ and apply DT to:

(i)*	$\alpha \rightarrow \beta$	[hypothesis]
(ii)	$\beta \equiv \gamma \,\&\, \beta$	[$\beta \supset \gamma$]
(iii)*	$\alpha \rightarrow \gamma \,\&\, \beta$	[RPE, ii, i]
(iv)*	$\alpha \rightarrow \gamma$	[A2, iii]

For B2, apply DT to:

(i)*	$\alpha \rightarrow \beta$	[hypothesis]
(ii)*	$\alpha \rightarrow \beta \vee \gamma$	[B1, $\beta \supset \beta \vee \gamma$, i]
(iii)	$\alpha \rightarrow \beta \vee \gamma$	[B0, $\gamma \supset \beta \vee \gamma$]
(iv)*	$\alpha \vee \gamma \rightarrow \beta \vee \gamma$	[A4, ii, iii]

For B3, apply DT to:

(i)*	$\alpha \rightarrow \gamma$	[hypothesis]
(ii)*	$\alpha \rightarrow \beta$	[hypothesis]
(iii)*	$\beta \rightarrow \alpha$	[hypothesis]
(iv)*	$\alpha \,\&\, \beta \rightarrow \gamma$	[A3, i, ii]

(v)*	$\alpha \,\&\, \beta \to (\alpha \supset \gamma)$	[B1, $\gamma \supset (\alpha \supset \gamma)$, iv]
(vi)	$\sim\alpha \,\&\, \beta \to (\alpha \supset \gamma)$	[B0, $\sim\alpha \,\&\, \beta \supset (\alpha \supset \gamma)$]
(vii)*	$[(\alpha \,\&\, \beta) \vee (\sim\alpha \,\&\, \beta)] \to (\alpha \supset \gamma)$	[A4, v, vi]
(viii)*	$\beta \to (\alpha \supset \gamma)$	[RPE, $\beta \equiv [(\alpha \,\&\, \beta) \vee (\sim\alpha \,\&\, \beta)]$, vii]
(ix)*	$\beta \to \alpha \,\&\, (\alpha \supset \gamma)$	[A1, iii, viii]
(x)*	$\beta \to \gamma$	[B1, $[\alpha \,\&\, (\alpha \supset \gamma)] \supset \gamma$, ix]

For B4, assume $\vdash \beta \supset \alpha$ and $\vdash \alpha \,\&\, \gamma \supset \beta$, and apply DT to:

(i)*	$\alpha \to \gamma$	[hypothesis]
(ii)	$\beta \to \alpha$	B0, $\beta \supset \alpha$
(iii)	$\alpha \to \alpha$	A0
(iv)*	$\alpha \to \alpha \,\&\, \gamma$	[A1, i, iii]
(v)*	$\alpha \to \beta$	[B1, $\alpha \,\&\, \gamma \supset \beta$, iv]
(vi)*	$\beta \to \gamma$	[B3, i, v, ii]

1 The proof One half of the completeness theorem $\lambda(\mathcal{M}) = \theta(\mathcal{J})$ is easy, for it is a tedious but entirely routine exercise to check that each axiom A0-A4 is valid in all models in $\mathcal{M}$, and that our rules preserve validity, thus proving the inclusion $\theta(\mathcal{J}) \subseteq \lambda(\mathcal{M})$. To prove the opposite inclusion we need to show that if ϕ is *consistent*, i.e., $\sim\phi \notin \theta(\mathcal{J})$, then there is a model in which it is satisfiable.

We will, in fact, construct a *finite* model for ϕ. Thus our construction would tell us—if we did not already know it from [1]—that $\theta(\mathcal{J})$ is decidable. (For to know whether or not a formula is a thesis, we need only search through all proofs and through all members of some representative set of finite models until we either find a proof for the formula or a model for its negation.)

For any $\alpha \in \mathcal{L}$, let $\kappa(\alpha)$ = the least k such that all variables in α are among $p_1, p_2, \ldots, p_k$. Let $\nu(\alpha)$ = the depth of nesting of arrows in α:

$\nu(p_i) = 0$
$\nu(\alpha) < n$ if α is a truth-functional compound of formulas with $\nu < n$
$\nu(\beta \to \gamma) = 1 + \max(\nu(\beta), \nu(\gamma))$.

Call α *prime* over a finite set $X \subseteq \mathcal{L}$ if $\alpha = \bigwedge A$, where for each $\xi \in X$, exactly one of ξ, $\sim\xi$ belongs to A. Define:

$$X_0^k = \{p_1, p_2, \ldots, p_k\}$$
$$Y_n^k = \{\alpha\colon \alpha \text{ consistent } \& \ \alpha \text{ prime over } X_n^k\}$$
$$Z_n^k = \{\bigvee A\colon A \subseteq Y_n^k\}$$
$$X_{n+1}^k = X_0^k \cup \{\psi \to \chi\colon \psi, \chi \in Z_n^k\}.$$

Readers of Carnap can think of each $\alpha \in Y_n^k$ as a 'state description' of sorts.

Lemma 0 *In the above notation, we have:*

(a) *A formula is provably equivalent to a truth-functional compound of elements of X_n^k iff it is provably equivalent to an element of Z_n^k.*
(b) *Any element of X_n^k is provably equivalent to an element of X_{n+1}^k.*
(c) *Any element of Z_n^k is provably equivalent to an element of Z_{n+1}^k.*
(d) *A formula is provably equivalent to a formula with $\kappa \leqslant k$ and $\nu \leqslant n$ iff it is provably equivalent to an element of Z_n^k.*

Proof: (a) Every element of Z_n^k *is* a truth-functional compound of elements of X_n^k; conversely, if such a compound is reduced to full disjunctive normal form and inconsistent disjuncts dropped, it belongs to Z_n^k. (b) is trivial for $n = 0$; using (a), (b) for $n = m$ implies (c) for $n = m$; using RPE, (c) for $n = m$ implies (b) for $n = m + 1$. So (b) and (c) hold for all n, and using them (d) is a routine exercise.

Thus any formula ϕ is provably equivalent to $\bigvee F$ for some $F \subseteq Y_{\nu(\phi)}^{\kappa(\phi)}$, with $F \neq \phi$ iff ϕ is consistent. Since any valuation assigns the same set to provably equivalent formulas, we are reduced to proving:

Proposition *For any n, any element of any Y_n^k is satisfiable in some model in $\mathcal{M}$.*

We proceed by induction on n, the case $n = 0$ being trivial. Assume the Proposition for $n = m$, and let $\phi \in Y_{m+1}^k$ be given, to prove ϕ satisfiable in some model in $\mathcal{M}$.

We reserve the letters α, β, γ, δ for elements of Y_m^k, and A, B, C, D, for subsets of Y_n^k. Distinct prime formulas (and, more generally, disjunctions of disjoint sets of prime formulas) are logically incompatible: If $\alpha \neq \beta$, $\vdash \sim(\alpha \,\&\, \beta)$; and if $A \cap B = \phi$, $\vdash \sim(\bigvee A \,\&\, \bigvee B)$. If $\alpha \notin B$, we write $\alpha \rhd B$ if $\vdash \phi \supset (\alpha \vee \bigvee B \rightarrow \bigvee B)$, and α/B otherwise.

Intuitively, $\alpha \rhd B$ may be thought of as saying that $\bigvee B$ is (assuming ϕ) a less remote possibility than α. On this reading of $\rhd$, what Lemmas 1 and 2 below say should be plausible enough.

Lemma 1 *Let $A \cap B = \phi$. Then $\vdash \phi \supset (\bigvee A \vee \bigvee B \rightarrow \bigvee B)$ iff for all $\alpha \in A$, $\alpha \rhd B$.*

Proof: Immediate from A4 and B4.

Lemma 2 *If α/B and $\alpha \rhd C$, then for some $\gamma \in C$, $\gamma/B \cup \{\alpha\}$.*

Proof: Assume for contradiction that α/B, $\alpha \rhd C$ and no suitable γ exists. Let $B' = B - C$, $C' = C - B$, $D = B \cap C$, so $\{\alpha\}$, B', C', D are pairwise disjoint. Consider the following deduction, in which (i) is simply the assumption $\alpha \rhd C$, while (ii) follows from the nonexistence of any suitable γ:

(i)	$\phi \supset (\alpha \vee \bigvee C' \vee \bigvee D \rightarrow \bigvee C' \vee \bigvee D)$	
(ii)	$\phi \supset (\alpha \vee \bigvee B' \vee \gamma \vee \bigvee D \rightarrow \alpha \vee \bigvee B' \vee \bigvee D)$	[all $\gamma \in C'$]
(iii)	$\phi \supset (\alpha \vee \bigvee B' \vee \bigvee C' \vee \bigvee D \rightarrow \alpha \vee \bigvee B' \vee \bigvee D)$	[A4, ii]
(iv)	$\phi \supset (\alpha \vee \bigvee B' \vee \bigvee C' \vee \bigvee D \rightarrow \bigvee B' \vee \bigvee C' \vee \bigvee D)$	[B2, i]
(v)	$(\alpha \vee \bigvee B' \vee \bigvee D) \,\&\, (\bigvee B' \vee \bigvee C' \vee \bigvee D) \supset (\bigvee B' \vee \bigvee D)$	[disjointedness]
(vi)	$\phi \supset (\alpha \vee \bigvee B' \vee \bigvee C' \vee \bigvee D \rightarrow \bigvee B' \vee \bigvee D)$	[A1, iii, iv, B1, v]
(vii)	$\phi \supset (\alpha \vee \bigvee B' \vee \bigvee D \rightarrow \bigvee B' \vee \bigvee D)$.	[B4, vi]

Here (vii) contradicts the assumption α/B.

Our ultimate aim is to build a model (W, R) containing a $w \in W$ at which ϕ is true. Any such model will have to contain for certain formulas γ some $x \in W$ for γ to be true at. And then for any C with $\gamma \rhd C$, there will have to be a $\delta \in C$ and a $y \in W$ with $Rwyx$ for δ to be true at. And then for any D with $\delta \rhd D$

there will have to be an $\eta \in D$ and a $z \in W$ with $Rwzy$ for η to be true at, etc. Therefore (W, R) will have to be fairly complicated, and so will the construction that builds it. In fact, it will have to be as complicated as Lemma 3 below.

For any α, B, C satisfying the hypotheses of Lemma 2, let us fix a γ satisfying the conclusion, and call it $\Gamma(\alpha, B, C)$. For any $\gamma \in Y^k_m$, let $r(\gamma)$ be the number of $C \subseteq Y^k_m$ such that $\gamma \rhd C$; fix an enumeration of these C's and for $j \leqslant r(\gamma)$ let $C_j(\gamma)$ be the j^{th} one.

Having this machinery, consider a pair such that α/B. For certain finite sequences s of natural numbers, we will define $\gamma^{\alpha,B,s}$ and $D^{\alpha,B,s}$ satisfying $\gamma^{\alpha,B,s}/D^{\alpha,B,s}$:

If $s = \phi$ (empty sequence), $D^{\alpha,B,s} = B$ and $\gamma^{\alpha,B,s} = \alpha$.

If $s = t \# j$ (t with j adjoined), D and γ will be defined for s iff they are defined for t and $r(\gamma^{\alpha,B,t}) \geqslant j$. In that case, $D^{\alpha,B,s} = D^{\alpha,B,t} \cup \{\gamma^{\alpha,B,t}\}$, and $\gamma^{\alpha,B,s} = \Gamma(\gamma^{\alpha,B,t}, D^{\alpha,B,t}, C_j(\gamma^{\alpha,B,t}))$.

Let $S^{\alpha,B} = \{s: \gamma^{\alpha,B,s} \text{ is defined}\}$. All one needs to remember is:

Lemma 3 *For any pair such that α/B:*

(a) *$S^{\alpha,B}$ is finite.*
(b) *If $s \in S^{\alpha,B}$ and $\gamma^{\alpha,B,s} \rhd C$, then there exists an $s' \in S^{\alpha,B}$ extending s, such that $\gamma^{\alpha,B,s'} \in C$.*
(c) *For any $s \in S^{\alpha,B}$, $\gamma^{\alpha,B,s} \notin B$.*

Proof: If s' extends s and $s' \in S^{\alpha,B}$, then $\gamma^{\alpha,B,s'} \neq \gamma^{\alpha,B,s}$, since the latter does, and the former does not, belong to $D^{\alpha,B,s'}$. Thus $S^{\alpha,B}$ contains no infinite subset linearly ordered by the relation 'extension of'. On the other hand, if $t \in S^{\alpha,B}$, then there are only finitely many j with $t \# j \in S^{\alpha,B}$. (a) follows by König's Infinity Lemma. (b) is plain from the construction. The $D^{\alpha,B,s}$ get bigger as s gets longer, so all include $D^{\alpha,B,\phi} = B$. But $\gamma^{\alpha,B,s} \notin D^{\alpha,B,s}$, so (c) follows.

Let now $Q = \{(\alpha, B, s): s \in S^{\alpha,B}\}$, which is finite by Lemma 3(a). We invoke the proposition for $n = m$ to obtain, for each $q \in Q$, a $(W^q, R^q) \in \mathcal{M}$, and a valuation V^q in it, and an $x^q \in V^q(\gamma^q)$. Without loss of generality, we assume the W^q disjoint. We paste these things together to obtain $(W^*, R^*) \in \mathcal{M}$, and a valuation V^* in it, and an $x^* \in V^*(\phi)$, thus proving the proposition for $n = m + 1$.

Let W^* be the union of the W^q plus one new element x^*. Let R^*xyz hold if *either* for some q we have $x, y, z \in W^q$ and R^qxyz, *or* if $x = x^*$ and for some pair satisfying α/B, y and z are respectively of the forms $x^{\alpha,B,s'}$ and $x^{\alpha,B,s}$ and s' extends s. (The *longer* sequence gives the *less* remote element.) Let $V^*(p_i) \cap W^q = V^q(p_i)$, it being possible to accomplish this for all q simultaneously since the W^q are disjoint. Let $x^* \in V^*(p_i)$ iff p_i is a conjunct of ϕ.

Unpacking the definitions, we see $V^*(\psi) \cap W^q = V^q(\psi)$ for *any* ψ. What we want to show is that $x^* \in V^*(\phi)$. Now $\phi \in Y^k_{m+1}$ is a conjunction of certain elements of X^k_{m+1} and of the negations of certain elements of X^k_{m+1}. ϕ being prime and consistent, $\xi \in X^k_{m+1}$ will be a conjunct of ϕ iff $\sim\xi$ is not a conjunct of ϕ iff $\vdash \phi \supset \xi$. To show $x^* \in V^*(\phi)$ it will suffice to show $x^* \in V^*(\xi)$ iff $\vdash \phi \supset \xi$, for all $\xi \in X^k_{m+1}$. Now the elements of X^k_{m+1} consist of variables, which have already been taken care of in the definition of V^*, and of formulas of form $\psi \to \chi$ with $\psi, \chi \in Z^k_m$.

Let us consider one such conditional, and set $\psi = \bigvee E$, $\chi = \bigvee F$, $C = E \cap F$, $D = E - F$, so $\psi \to \chi$ amounts to $\bigvee C \vee \bigvee D \to \bigvee C$. Now on the one hand, Lemma 1 tells us that $\vdash \phi \supset (\bigvee C \vee \bigvee D \to \bigvee C)$ iff:

(I) $\delta \rhd C$ for all $\delta \in D$.

While on the other hand, the finiteness of $W^*_{x^*}$ (= $\{x^q: q \in Q\}$) means that the definition of $x^* \in V^*(\bigvee C \vee \bigvee D \to \bigvee C)$ boils down to:

(II) $\forall y \in W^*_{x^*} \cap V(\bigvee D) \exists z \in W^*_{x^*} \cap V(\bigvee C)(R^* x^* zy)$.

Now for $y = x^q$, $z = x^{q'}$ in $W^*_{x^*}$, we have $y \in V^q(\gamma^q) \subseteq V^*(\gamma^q)$. Thus $y \in V^*(\bigvee D)$ iff $\gamma^q \in D$, and similarly $z \in V^*(\bigvee C)$ iff $\gamma^{q'} \in C$. Moreover, if $q = (\alpha, B, s)$, $q' = (\alpha', B', s')$, then $R^* x^* zy$ iff $\alpha = \alpha'$, $B = B'$, and s' extends s. Thus (II) amounts to:

(II′) $\forall (\alpha, B, s) \in Q[\gamma^{\alpha, B, s} \in D \supset \exists (\alpha, B, s') \in Q(\gamma^{\alpha, B, s'} \in C \;\&\; s' \text{ extends } s)]$.

To complete the proof, we must show (I) equivalent to (II′). Well, (I) implies (II′) by Lemma 3(b). And if $\delta \in D$ is a counterexample to (I), setting $\alpha = \delta$, $B = C$, $s = \phi$ provides a counterexample to (II′) by Lemma 3(c).

3 ***Variants*** We next indicate for various model-classes $\mathcal{N} \subseteq \mathcal{M}$ obtained by imposing characteristic restrictions on R, what axioms we need to add to $\mathcal{S}$ to obtain a $\mathcal{T}$ with $\lambda(\mathcal{N}) = \theta(\mathcal{T})$. Many of these results appear in the Appendix to [2] (others appear scattered through the literature in various disguises).

We can start with a degenerate case, *antisymmetry*:

$$\forall x \in W \,\forall y, z \in W_x (Rxyz \;\&\; Rxzy \supset y = z).$$

It turns out that we do not need to add any new axioms to handle this, for on close inspection the construction of the last section is seen to produce an antisymmetric model.

Less trivial are the following:

(C′) connectivity: $\forall x \in W \,\forall y, z \in W_x (Rxyz \vee Rxzy)$
(C″) connectivity *plus* antisymmetry
(C_0) nonvacuity: $\forall x \in W \,(W_x \neq \phi)$
(C_1) centrality: $\forall x \in W \,(x \in W_x \;\&\; \forall y \in W_x (y \neq x \supset Rxxy \;\&\; {\sim}Rxyx))$.

We write, e.g., $\mathcal{M}''$ for the class of models satisfying (C″), $\mathcal{M}_1$ for those satisfying (C_1), $\mathcal{M}'_0$ for those satisfying (C′) and (C_0).

To handle these we will need some new axioms:

(D′) $(p \vee q \to {\sim}p) \supset (p \vee r \to {\sim}p) \vee (q \vee r \to {\sim}r)$
(D″) $(p \vee q \to p) \vee (p \vee q \to q)$
(D_0) ${\sim}(\top \to \bot)$
(D_1) (a) $p \;\&\; q \supset (p \to q)$
(b) $(p \to q) \supset (p \supset q)$.

We write, e.g., $\mathcal{S}''$ for $\mathcal{S}$ + (D″), $\mathcal{S}_1$ for $\mathcal{S}$ + (D_1)(a) + (D_1)(b), $\mathcal{S}'_0$ for $\mathcal{S}$ + (D′) + (D_0).

Then in every case, if $\mathcal{M}$ is a model-class and $\mathcal{T}$ a standard system bearing similar indices, then $\lambda(\mathcal{M}) = \theta(\mathcal{T})$. As an example, we will outline the modifications of the construction presented in the last section needed to prove $\lambda(\mathcal{M}') = \theta(\mathcal{S}')$. We leave it to the reader to verify that (D$'$) is indeed valid in all connected models, thus establishing $\theta(\mathcal{S}') \subseteq \lambda(\mathcal{M}')$. For the opposite inclusion we need a thesis of $\mathcal{S}'$:

(B5) $\vdash (p \vee q \vee r \rightarrow \sim p) \supset (p \vee q \rightarrow \sim p) \vee (p \vee r \rightarrow \sim p)$.

For a justification in the style of B0-B4 above, it will suffice to deduce from (i)-(iii) below a conclusion contradicting (ii):

(i)	$p \vee q \vee r \rightarrow \sim p$	[hypothesis]
(ii)	$\sim(p \vee q \rightarrow \sim p)$	[hypothesis]
(iii)	$\sim(p \vee r \rightarrow \sim p)$	[hypothesis]
(iv)	$(p \vee q \rightarrow \sim p) \vee (q \vee r \rightarrow \sim q)$	[D$'$, i]
(v)	$(p \vee r \rightarrow \sim p) \vee (q \vee r \rightarrow \sim r)$	[D$'$, i]
(vi)	$(q \vee r \rightarrow \sim q) \,\&\, (q \vee r \rightarrow \sim r)$	[ii, iii, iv, v]
(vii)	$q \vee r \rightarrow \sim q \,\&\, \sim r$	[A1, vi]
(viii)	$p \vee q \vee r \rightarrow p \vee (\sim q \,\&\, \sim r)$	[B2, vii]
(ix)	$p \vee q \vee r \rightarrow p \,\&\, [p \vee (\sim q \,\&\, \sim r)]$	[A1, i, viii]
(x)	$p \vee q \vee r \rightarrow \sim(p \vee q \vee r)$	[RPE, ix]
(xi)	$p \vee q \rightarrow \sim(p \vee q \vee r)$	[B4, x]
(xii)	$p \vee q \rightarrow \sim p$	[B1, xi]

Having this, we imitate the construction of the last section, beginning with the introduction of the X's, Y's, and Z's (understanding 'consistency' in the definition of Y as relative to $\mathcal{S}'$ rather than $\mathcal{S}$, of course). We assume every $\alpha \in Y^k_m$ is satisfiable in a connected model, and consider $\phi \in Y^k_{m+1}$. In addition to $\triangleright$ and / we define $Q = \{\alpha \in Y^k_m\text{: not } \vdash \phi \supset (\alpha \rightarrow \perp)\}$ and write $\alpha \precsim \beta$ if $\alpha, \beta \in Q$ and not $\vdash \phi \supset (\alpha \vee \beta \rightarrow \sim \alpha)$.

Lemma 1 is still valid, and (B5) provides a supplement:

Lemma 1$'$ *Let $A \cap B = \phi$. Then $\vdash \phi \supset (\bigvee A \vee \bigvee B \rightarrow \bigvee B)$ iff for some $\beta \in B$, $\vdash \phi \supset (\bigvee A \vee \beta \rightarrow \beta)$.*

Lemma 2 is still valid. Note that $\alpha \in Q$ iff α/Q iff $\exists B(\alpha/B)$. So Lemma 2 tells us: If $\alpha \in Q$ and $\alpha \triangleright C$, then $\gamma \precsim \alpha$ for some $\gamma \in C$.

Lemma 3 of the last section is, happily, not needed. But we have:

Lemma 3$'$ *The relation $\precsim$ is reflexive, transitive, and connected.*

Here (D$'$) is just what is needed to give transitivity, and the other two are easy.

Much as in the last section we invoke our induction hypothesis to obtain for each $\alpha \in Q$ a connected (W^α, R^α), and a valuation V^α in it, and an $x^\alpha \in V^\alpha(\alpha)$. To paste these together, let W^* be the union of the W^α plus one new element x^*. Let R^*xyz hold if *either* for some α we have $x, y, z \in W^\alpha$ and $R^\alpha xyz$, *or* if $x = x^*$ and for some α and β we have $y = x^\alpha$ and $z = x^\beta$ and $\alpha \precsim \beta$. Let $V^*(p_i) \cap W^\alpha = V^\alpha(p_i)$, with $x^* \in V^*(p_i)$ iff p_i is a conjunct of ϕ.

We finally need to show that for disjoint $C, D \subseteq Y^k_m$, that $\vdash \phi \supset (\bigvee C \vee \bigvee D \rightarrow \bigvee C)$ iff $x^* \in V^*(\bigvee C \vee \bigvee D \rightarrow \bigvee C)$. But this task will be left to the interested reader.

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