Material for the Final Exam: Answers

Final Exam Date: Wednesday, Dec 16, 8:00am - 10:00am, LEF 2205

Part 1: Basic Concepts

Fill in the blank

1. A ___sound___ argument is one that has all true premises, and whose premises support the conclusion.

2. A formula of sentential logic is called ___contradictory___ just in case it is false on every truth-value assignment.

3. A formula of sentential logic is called ___a tautology___ just in case it is true on every truth-value assignment.

4. A formula of sentential logic is called ___contingent___ just in case it is true on some truth-value assignments, and false on others.

5. Two formulae of sentential logic are logically equivalent if and only if they are assigned ___the same___ truth-value(s) on every truth-value assignment.

6. An interpretation includes each of the following:

   (a) A ___domain___ .

   (b) An assignment to each individual constant of an individual from the domain of discourse.

   (c) An assignment of truth-values to 0-place predicate letters.

   (d) For $n > 0$, an assignment of a set of ordered n-tuples of members of the domain of discourse to each n-place predicate letter.

7. $\forall$ and $\exists$ are called ___quantifier symbols (or quantifiers)___ .

8. If $(\varphi \rightarrow \psi)$ is a tautology, then ψ is a ___logical consequence___ of φ.

9. If $(\varphi \leftrightarrow \psi)$ is a tautology, then φ and ψ are ___logically equivalent___ .

10. Provide the symbol we use for each of the following connectives:

    (a) Conjunction: ___&___ .

    (b) Disjunction: ___$\vee$___.

    (c) Conditional: ___$\rightarrow$___.

    (d) Negation: ___$\neg$___.

    (e) Bi-conditional (if, and only if): ___$\leftrightarrow$___.

Part 2: Valid/Sound Arguments

True or False? (Circle your answer)

1. If an argument has a false conclusion it is invalid.

   True False

2. No unsound arguments have a true conclusion.

   True False

3. If it is not possible for the conclusion of an argument to be false, then the argument is valid.

   True False

4. Every invalid argument has a false conclusion.

   True False

5. Some invalid arguments have a false conclusion.

   True False

6. All sound arguments are valid.

   True False

7. If an argument has true premises and a true conclusion, then it is sound.

   True False

8. If the conclusion of a valid argument is false, then at least one of its premises is false.

   True False

9. Some invalid arguments have a false premise.

   True False

10. No sound arguments have a false conclusion.

    True False

11. Some invalid arguments have a true conclusion.

    True False

12. Every invalid argument has a true conclusion.

True False

13. A valid argument with twenty true premises and one false premise is more sound than an argument with three true premises and one false one.

True False

14. Some valid arguments are unsound.

True False

15. If the conclusion of a valid argument is false, then all of its premises are false as well.

True False

16. If the conclusion of a valid argument is true, the premises must be true as well.

True False

17. If an argument is sound, then its conclusion follows from its premises.

True False

18. All unsound arguments are invalid.

True False

19. Some valid arguments have a true conclusion.

True False

20. Every sound argument has a true conclusion.

True False

21. If an argument has a conclusion that is logically equivalent to one of its premises, then the argument is valid.

True False

22. If φ and ψ are logically equivalent, then $\neg\varphi$ and $\neg\psi$ are logically equivalent.

True False

23. If an argument has two premises that are logically equivalent, then the argument is valid.

True False

24. If an argument has the sentence ‘either it is raining or it is not raining’ as one of its premises, then the argument is valid.

True   False

25. If an argument has the sentence ‘either it is raining or it is not raining’ as its conclusion, then the argument is valid.

True   False

Part 3: Translations

Translate the following sentences into predicate logic using the following translation key:

$S(x)$	x is a student.
$P(x)$	x is a professor.
$R(x, y)$	x respects y

(See also the translation problems handed out earlier in the semester).

1. Every professor is respected by some student or other.

$$(\forall x)(P(x) \to \exists(y)(S(y) \;\&\; R(y, x)))$$

2. Every professor respects some student or other.

$$(\forall x)(\exists y)(P(x) \;\&\; S(y) \;\&\; R(x, y))$$

3. Every student is respected by some professor or other.

$$(\forall x)(S(x) \to (\exists y)(P(y) \;\&\; R(y, x)))$$

4. For every professor, there is a student who doesn’t respect that professor.

$$(\forall x)(\exists y)(P(x) \;\&\; S(y) \;\&\; \neg R(y, x))$$

5. For every student, there is a professor who doesn’t respect that student.

$$(\forall x)(\exists y)(S(x) \;\&\; P(y) \;\&\; \neg R(y, x))$$

6. For every professor, there is a student whom the professor doesn’t respect.

$$(\forall x)(\exists y)(P(x) \;\&\; S(y) \;\&\; \neg R(x, y))$$

7. There is a student who respects every professor.

$$(\exists x)(S(x) \mathbin{\&} (\forall y)(P(y) \rightarrow R(x, y)))$$

8. There is a student who is respected by every professor.

$$(\exists x)(S(x) \mathbin{\&} (\forall y)(P(y) \rightarrow R(y, x)))$$

9. There is no professor who respects every student.

$$\neg(\exists x)(P(x) \mathbin{\&} (\forall y)(S(y) \rightarrow R(x, y)))$$

10. There is no student who respects no professor.

$$\neg(\exists x)(S(x) \mathbin{\&} \neg(\exists y)(P(y) \mathbin{\&} R(x, y)))$$

11. There is a student who does not respect every professor.

$$(\exists x)(S(x) \mathbin{\&} \neg(\forall y)(P(y) \rightarrow R(x, y)))$$

12. There is a professor who is not respected by every student.

$$(\exists x)(P(x) \mathbin{\&} \neg(\forall y)(S(y) \rightarrow R(y, x)))$$

13. There is no student who doesn't respect at least one professor.

$$\neg(\exists x)(S(x) \mathbin{\&} \neg(\exists y)(P(y) \mathbin{\&} R(x, y)))$$

14. There is no professor who doesn't respect at least one student.

$$\neg(\exists x)(P(x) \mathbin{\&} \neg(\exists y)(S(y) \mathbin{\&} R(x, y)))$$

15. There is a professor who doesn't respect any student who doesn't respect every professor.

$$(\exists x)(P(x) \mathbin{\&} \neg(\exists y)((R(x, y) \mathbin{\&} S(y)) \mathbin{\&} \neg(\exists z)(P(z) \mathbin{\&} \neg R(y, z))))$$

Part 4: Truth-Tables

For each of the formulas, use a truth-table to determine if the formula is a tautology, contradictory or a contingent formula. You must explain you answer.

1. $(P \,\&\, Q) \rightarrow \neg(P \vee Q)$

   Solution. The truth-table for $(P \,\&\, Q) \rightarrow \neg(P \vee Q)$ is:

P	Q	$P \,\&\, Q$	$P \vee Q$	$\neg(P \vee Q)$	$(P \,\&\, Q) \rightarrow \neg(P \vee Q)$
T	T	T	T	F	F
T	F	F	T	F	T
F	T	F	T	F	T
F	F	F	F	T	T

   Since there are some truth-value assignments that make $(P \,\&\, Q) \rightarrow \neg(P \vee Q)$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

   (a) When P is T, Q is T, then $(P \,\&\, Q) \rightarrow \neg(P \vee Q)$ is F.

2. $((P \,\&\, Q) \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$

   Solution. The truth-table for $((P \,\&\, Q) \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$ is:

P	Q	R	$P \,\&\, Q$	$(P \,\&\, Q) \rightarrow R$	$Q \rightarrow R$	$P \rightarrow (Q \rightarrow R)$	$((P \,\&\, Q) \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

   Since every truth-value assignment makes $((P \,\&\, Q) \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$ true (there are only 'T' in the column underneath the formula), it is a **tautology**.

3. $(P \vee Q) \rightarrow P$

   Solution. The truth-table for $(P \vee Q) \rightarrow P$ is:

P	Q	$P \vee Q$	$(P \vee Q) \rightarrow P$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Since there are some truth-value assignments that make $(P \vee Q) \to P$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

(a) When P is F, Q is T, then $(P \vee Q) \to P$ is F.

4. $(P \,\&\, (P \to Q)) \to \neg Q$

Solution. The truth-table for $(P \,\&\, (P \to Q)) \to \neg Q$ is:

P	Q	$P \to Q$	$P \,\&\, (P \to Q)$	$\neg Q$	$(P \,\&\, (P \to Q)) \to \neg Q$
T	T	T	T	F	F
T	F	F	F	T	T
F	T	T	F	F	T
F	F	T	F	T	T

Since there are some truth-value assignments that make $(P \,\&\, (P \to Q)) \to \neg Q$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

(a) When P is T, Q is T, then $(P \,\&\, (P \to Q)) \to \neg Q$ is F.

5. $(P \leftrightarrow Q) \to (P \,\&\, Q)$

Solution. The truth-table for $(P \leftrightarrow Q) \to (P \,\&\, Q)$ is:

P	Q	$P \leftrightarrow Q$	$P \,\&\, Q$	$(P \leftrightarrow Q) \to (P \,\&\, Q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	F

Since there are some truth-value assignments that make $(P \leftrightarrow Q) \to (P \,\&\, Q)$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

(a) When P is F, Q is F, then $(P \leftrightarrow Q) \to (P \,\&\, Q)$ is F.

6. $(P \to Q) \vee (Q \to P)$

Solution. The truth-table for $(P \to Q) \vee (Q \to P)$ is:

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Since every truth-value assignment makes $(P \to Q) \vee (Q \to P)$ true (there are only 'T' in the column underneath the formula), it is a **tautology**.

7. $P \to (Q \vee R)$

   Solution. The truth-table for $P \to (Q \vee R)$ is:

P	Q	R	$Q \vee R$	$P \to (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

   Since there are some truth-value assignments that make $P \to (Q \vee R)$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

   (a) When P is T, Q is F, R is F, then $P \to (Q \vee R)$ is F.

8. $(\neg(Q \vee R) \,\&\, Q) \to P$

   Solution. The truth-table for $(\neg(Q \vee R) \,\&\, Q) \to P$ is:

Q	R	P	$Q \vee R$	$\neg(Q \vee R)$	$\neg(Q \vee R) \,\&\, Q$	$(\neg(Q \vee R) \,\&\, Q) \to P$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	T	F	F	T
T	F	F	T	F	F	T
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

   Since every truth-value assignment makes $(\neg(Q \vee R) \,\&\, Q) \to P$ true (there are only 'T' in the column underneath the formula), it is a **tautology**.

9. $(P \to Q) \,\&\, (Q \to P)$

   Solution. The truth-table for $(P \to Q) \,\&\, (Q \to P)$ is:

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \,\&\, (Q \to P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Since there are some truth-value assignments that make $(P \to Q) \& (Q \to P)$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

(a) When P is T, Q is F, then $(P \to Q) \& (Q \to P)$ is F.

(b) When P is F, Q is T, then $(P \to Q) \& (Q \to P)$ is F.

10. $(P \& Q) \vee (P \& \neg Q)$

Solution. The truth-table for $(P \& Q) \vee (P \& \neg Q)$ is:

P	Q	$P \& Q$	$\neg Q$	$P \& \neg Q$	$(P \& Q) \vee (P \& \neg Q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	F	F	F
F	F	F	T	F	F

Since there are some truth-value assignments that make $(P \& Q) \vee (P \& \neg Q)$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

(a) When P is F, Q is T, then $(P \& Q) \vee (P \& \neg Q)$ is F.

(b) When P is F, Q is F, then $(P \& Q) \vee (P \& \neg Q)$ is F.

11. $\neg(P \& \neg P)$

Solution. The truth-table for $\neg(P \& \neg P)$ is:

P	$\neg P$	$P \& \neg P$	$\neg(P \& \neg P)$
T	F	F	T
F	T	F	T

Since every truth-value assignment makes $\neg(P \& \neg P)$ true (there are only 'T' in the column underneath the formula), it is a **tautology**.

12. $\neg(P \vee \neg P)$

Solution. The truth-table for $\neg(P \vee \neg P)$ is:

P	$\neg P$	$P \vee \neg P$	$\neg(P \vee \neg P)$
T	F	T	F
F	T	T	F

Since every truth-value assignment makes $\neg(P \vee \neg P)$ false (there are only 'F' in the column underneath the formula), it is a **contraduction**.

13. $\neg(P \,\&\, Q) \vee (\neg P \vee \neg Q)$

Solution. The truth-table for $\neg(P \,\&\, Q) \vee (\neg P \vee \neg Q)$ is:

P	Q	$P \,\&\, Q$	$\neg(P \,\&\, Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(P \,\&\, Q) \vee (\neg P \vee \neg Q)$
T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Since there are some truth-value assignments that make $\neg(P \,\&\, Q) \vee (\neg P \vee \neg Q)$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

(a) When P is T, Q is T, then $\neg(P \,\&\, Q) \vee (\neg P \vee \neg Q)$ is F.

14. $\neg(P \vee Q) \vee (\neg P \,\&\, \neg Q)$

Solution. The truth-table for $\neg(P \vee Q) \vee (\neg P \,\&\, \neg Q)$ is:

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \,\&\, \neg Q$	$\neg(P \vee Q) \vee (\neg P \,\&\, \neg Q)$
T	T	T	F	F	F	F	F
T	F	T	F	F	T	F	F
F	T	T	F	T	F	F	F
F	F	F	T	T	T	T	T

Since there are some truth-value assignments that make $\neg(P \vee Q) \vee (\neg P \,\&\, \neg Q)$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

(a) When P is T, Q is T, then $\neg(P \vee Q) \vee (\neg P \,\&\, \neg Q)$ is F.
(b) When P is T, Q is F, then $\neg(P \vee Q) \vee (\neg P \,\&\, \neg Q)$ is F.
(c) When P is F, Q is T, then $\neg(P \vee Q) \vee (\neg P \,\&\, \neg Q)$ is F.

15. $(P \vee Q) \vee (\neg P \,\&\, \neg Q)$

Solution. The truth-table for $(P \vee Q) \vee (\neg P \,\&\, \neg Q)$ is:

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \,\&\, \neg Q$	$(P \vee Q) \vee (\neg P \,\&\, \neg Q)$
T	T	T	F	F	F	T
T	F	T	F	T	F	T
F	T	T	T	F	F	T
F	F	F	T	T	T	T

Since every truth-value assignment makes $(P \vee Q) \vee (\neg P \,\&\, \neg Q)$ true (there are only 'T' in the column underneath the formula), it is a **tautology**.

16. $\neg(P \vee (P \mathbin{\&} Q))$

Solution. The truth-table for $\neg(P \vee (P \mathbin{\&} Q))$ is:

P	Q	$P \mathbin{\&} Q$	$P \vee (P \mathbin{\&} Q)$	$\neg(P \vee (P \mathbin{\&} Q))$
T	T	T	T	F
T	F	F	T	F
F	T	F	F	T
F	F	F	F	T

Since there are some truth-value assignments that make $\neg(P \vee (P \mathbin{\&} Q))$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

(a) When P is T, Q is T, then $\neg(P \vee (P \mathbin{\&} Q))$ is F.

(b) When P is T, Q is F, then $\neg(P \vee (P \mathbin{\&} Q))$ is F.

17. $((P \to Q) \to P) \to P$

Solution. The truth-table for $((P \to Q) \to P) \to P$ is:

P	Q	$P \to Q$	$(P \to Q) \to P$	$((P \to Q) \to P) \to P$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Since every truth-value assignment makes $((P \to Q) \to P) \to P$ true (there are only 'T' in the column underneath the formula), it is a **tautology**.

18. $(P \leftrightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow R)$

Solution. The truth-table for $(P \leftrightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow R)$ is:

P	Q	R	$Q \leftrightarrow R$	$P \leftrightarrow (Q \leftrightarrow R)$	$P \leftrightarrow Q$	$(P \leftrightarrow Q) \leftrightarrow R$	$(P \leftrightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	F	F	F	F	T
T	F	F	T	T	F	T	T
F	T	T	T	F	F	F	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	T	F	T	F	T

Since every truth-value assignment makes $(P \leftrightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow R)$ true (there are only 'T' in the column underneath the formula), it is a **tautology**.

19. $(P \vee Q) \rightarrow (P \,\&\, Q)$

Solution. The truth-table for $(P \vee Q) \rightarrow (P \,\&\, Q)$ is:

P	Q	$P \vee Q$	$P \,\&\, Q$	$(P \vee Q) \rightarrow (P \,\&\, Q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Since there are some truth-value assignments that make $(P \vee Q) \rightarrow (P \,\&\, Q)$ true and some that make it false, it is a **contingent** formula. The counterexamples are:

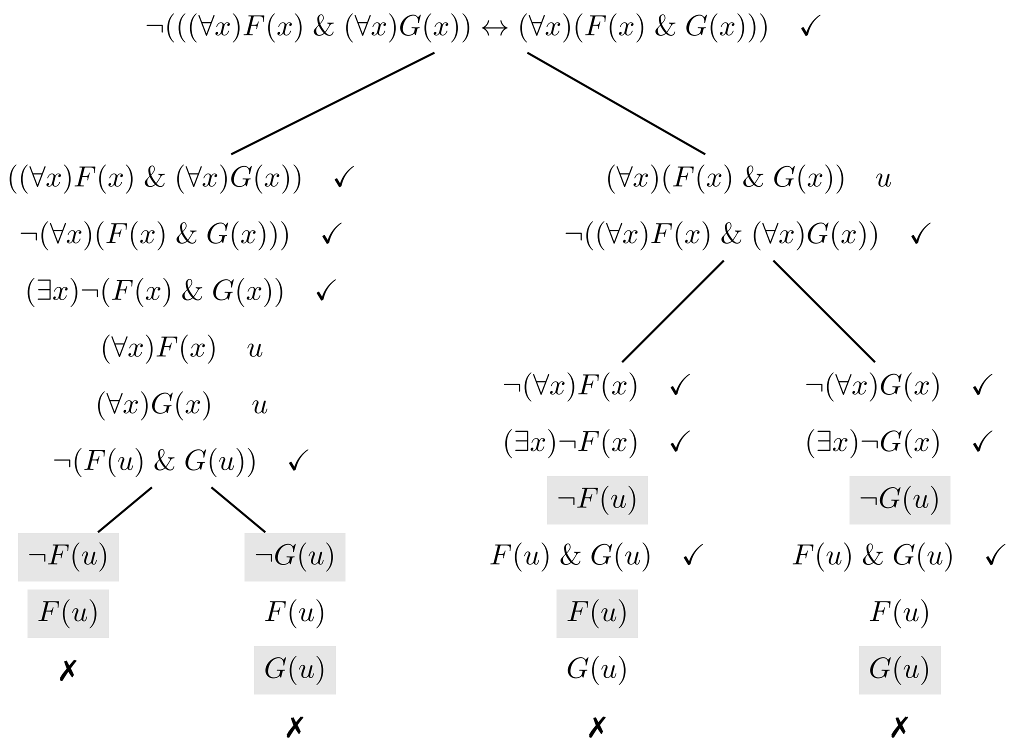
(a) When P is T, Q is F, then $(P \vee Q) \rightarrow (P \,\&\, Q)$ is F.

(b) When P is F, Q is T, then $(P \vee Q) \rightarrow (P \,\&\, Q)$ is F.

Part 5: Truth-Trees

For each of the following formulas, use a truth-tree to determine if it is a tautology. If it is not a tautology, provide an interpretation that makes the formula false.

1. $((\forall x)F(x) \,\&\, (\forall x)G(x)) \leftrightarrow (\forall x)(F(x) \,\&\, G(x))$



Since all of the branches are closed, the formula is a tautology.

2. $((\forall x)F(x) \vee (\forall x)G(x)) \rightarrow (\forall x)(F(x) \vee G(x))$

$$\neg(((\forall x)F(x) \vee (\forall x)G(x)) \rightarrow (\forall x)(F(x) \vee G(x))) \quad \checkmark$$

$$((\forall x)F(x) \vee (\forall x)G(x)) \quad \checkmark$$

$$\neg(\forall x)(F(x) \vee G(x)) \quad \checkmark$$

$$(\exists x)\neg(F(x) \vee G(x)) \quad \checkmark$$

$$\neg(F(u) \vee G(u)) \quad \checkmark$$

$$\neg F(u)$$

$$\neg G(u)$$

$$\begin{array}{cc} (\forall x)F(x) \quad u & (\forall x)G(x) \quad u \\ F(u) & G(u) \\ \text{✗} & \text{✗} \end{array}$$

Since all of the branches are closed, the formula is a tautology.

3. $(\forall x)(F(x) \vee G(x)) \to ((\forall x)F(x) \vee (\forall x)G(x))$

$$\neg((\forall x)(F(x) \vee G(x)) \to ((\forall x)F(x) \vee (\forall x)G(x))) \quad \checkmark$$

$$(\forall x)(F(x) \vee G(x)) \quad u \; v$$

$$\neg(((\forall x)F(x) \vee (\forall x)G(x))) \quad \checkmark$$

$$\neg(\forall x)F(x) \quad \checkmark$$

$$\neg(\forall x)G(x) \quad \checkmark$$

$$(\exists x)\neg F(x) \quad \checkmark$$

$$(\exists x)\neg G(x) \quad \checkmark$$

$$\neg F(u)$$

$$\neg G(v)$$

$$F(u) \vee G(u) \quad \checkmark$$

$$F(v) \vee G(v) \quad \checkmark$$

$$\begin{array}{ccc} F(u) & & G(u) \\ ✗ & & \begin{array}{cc} F(v) & G(v) \\ \bigcirc & ✗ \end{array} \end{array}$$

The is an open branch, so this formula is not a tautology. The interpretation with domain $D = \{\mathbf{U}, \mathbf{V}\}$ and

F	$\{\langle \mathbf{V} \rangle\}$
G	$\{\langle \mathbf{U} \rangle\}$

makes $(\forall x)(F(x) \vee G(x)) \to ((\forall x)F(x) \vee (\forall x)G(x))$ false.

4. $\neg(\exists x)F(x) \to (\forall x)(F(x) \to G(x))$

$$\neg(\neg(\exists x)F(x) \to (\forall x)(F(x) \to G(x))) \quad \checkmark$$
$$\neg(\exists x)F(x) \quad \checkmark$$
$$\neg(\forall x)(F(x) \to G(x)) \quad \checkmark$$
$$(\forall x)\neg F(x) \quad u$$
$$(\exists x)\neg(F(x) \to G(x)) \quad \checkmark$$
$$\neg(F(u) \to G(u)) \quad \checkmark$$
$$F(u)$$
$$\neg G(u)$$
$$\neg F(u)$$
$$\text{✗}$$

Since all of the branches are closed, the formula is a tautology.

5. $(\forall x)(F(x) \to G(x)) \to ((\exists x)F(x) \to (\exists x)G(x))$

$$\neg((\forall x)(F(x) \to G(x)) \to ((\exists x)F(x) \to (\exists x)G(x))) \quad \checkmark$$
$$(\forall x)(F(x) \to G(x)) \quad u$$
$$\neg((\exists x)F(x) \to (\exists x)G(x)) \quad \checkmark$$
$$(\exists x)F(x) \quad \checkmark$$
$$\neg(\exists x)G(x) \quad \checkmark$$
$$(\forall x)\neg G(x) \quad u$$
$$F(u)$$
$$F(u) \to G(u) \quad \checkmark$$
$$\begin{array}{cc} \swarrow & \searrow \\ \neg F(u) & G(u) \\ \text{✗} & \neg G(u) \\ & \text{✗} \end{array}$$

Since all of the branches are closed, the formula is a tautology.

6. $(\exists x)(F(x) \vee G(x)) \rightarrow ((\exists x)F(x) \vee (\exists x)G(x))$

$$\neg((\exists x)(F(x) \vee G(x)) \rightarrow ((\exists x)F(x) \vee (\exists x)G(x))) \quad \checkmark$$
$$(\exists x)(F(x) \vee G(x)) \quad \checkmark$$
$$\neg((\exists x)F(x) \vee (\exists x)G(x)) \quad \checkmark$$
$$\neg(\exists x)F(x) \quad \checkmark$$
$$\neg(\exists x)G(x) \quad \checkmark$$
$$(\forall x)\neg F(x) \quad u$$
$$(\forall x)\neg G(x) \quad u$$
$$F(u) \vee G(u) \quad \checkmark$$
$$\begin{array}{ccc} F(u) & \qquad & G(u) \\ \neg F(u) & & \neg F(u) \\ ✗ & & \neg G(u) \\ & & ✗ \end{array}$$

Since all of the branches are closed, the formula is a tautology.

7. $(\exists x)(\forall y)R(x,y) \rightarrow (\forall x)(\exists y)R(y,x)$

$$\neg((\exists x)(\forall y)R(x,y) \rightarrow (\forall x)(\exists y)R(y,x)) \quad \checkmark$$
$$(\exists x)(\forall y)R(x,y) \quad \checkmark$$
$$\neg(\forall x)(\exists y)R(y,x) \quad \checkmark$$
$$(\exists x)\neg(\exists y)R(y,x) \quad \checkmark$$
$$\neg(\exists y)R(y,u) \quad \checkmark$$
$$(\forall y)\neg R(y,u) \quad v$$
$$(\forall y)R(v,y) \quad u$$
$$\neg R(v,u)$$
$$R(v,u)$$
$$✗$$

Since all of the branches are closed, the formula is a tautology.

8. $(\forall x)(\exists y)R(x,y) \to (\exists y)(\forall x)R(x,y)$

$$\neg((\forall x)(\exists y)R(x,y) \to (\exists y)(\forall x)R(x,y)) \quad \checkmark$$
$$(\forall x)(\exists y)R(x,y) \quad u \; z_1 \; \cdots$$
$$\neg(\exists y)(\forall x)R(x,y) \quad \checkmark$$
$$(\forall y)\neg(\forall x)R(x,y) \quad u \; z_1 \; \cdots$$
$$\neg(\forall x)R(x,u) \quad \checkmark$$
$$(\exists x)\neg R(x,u) \quad \checkmark$$
$$\neg R(z_1,u)$$
$$(\exists y)R(u,y) \quad \checkmark$$
$$R(u,z_2)$$
$$(\exists y)R(z_1,y) \quad \checkmark$$
$$R(z_1,z_3)$$
$$\neg(\forall x)R(z_1,y) \quad \checkmark$$
$$(\exists x)\neg R(z_1,y) \quad \checkmark$$
$$\neg R(z_1,z_4)$$
$$\vdots$$

Since the branch is not closed, the formula is a tautology. Note that the branch is not open either (since it is not closed). However, we can find an interpretation that makes the formula false.

Suppose that the domain is $D = \{\mathbf{A}, \mathbf{B}\}$ and

R	$\{\langle \mathbf{A}, \mathbf{B}\rangle, \langle \mathbf{B}, \mathbf{A}\rangle\}$

This interpretation makes $(\forall x)(\exists y)R(x,y) \to (\exists y)(\forall x)R(x,y)$ false.

9. $(\exists x)\neg(\exists y)R(x,y) \to (\forall x)(\exists y)\neg R(y,x)$

$$\neg((\exists x)\neg(\exists y)R(x,y) \to (\forall x)(\exists y)\neg R(y,x)) \quad \checkmark$$
$$(\exists x)\neg(\exists y)R(x,y) \quad \checkmark$$
$$\neg(\forall x)(\exists y)\neg R(y,x) \quad \checkmark$$
$$(\exists x)\neg(\exists y)\neg R(y,x) \quad \checkmark$$
$$\neg(\exists y)\neg R(y,z_1) \quad \checkmark$$
$$(\forall y)\neg\neg R(y,z_1) \quad z_2$$
$$\neg(\exists y)R(z_2,y) \quad \checkmark$$
$$(\forall y)\neg R(z_2,y) \quad z_1$$
$$\neg\neg R(z_2,z_1) \quad \checkmark$$
$$R(z_2,z_1)$$
$$\neg R(z_2,z_1)$$
✗

Since all of the branches are closed, the formula is a tautology.

10. $(\exists y)(\forall x)R(x,y) \to (\forall x)(\exists y)R(x,y)$

$$\neg((\exists y)(\forall x)R(x,y) \to (\forall x)(\exists y)R(x,y)) \quad \checkmark$$
$$(\exists y)(\forall x)R(x,y) \quad \checkmark$$
$$\neg(\forall x)(\exists y)R(x,y) \quad \checkmark$$
$$(\exists x)\neg(\exists y)R(x,y) \quad \checkmark$$
$$(\forall x)R(x,z_1) \quad z_2$$
$$\neg(\exists y)R(z_2,y) \quad \checkmark$$
$$(\forall y)\neg R(z_2,y) \quad z_1$$
$$R(z_2,z_1)$$
$$\neg R(z_2,z_1)$$
✗

Since all of the branches are closed, the formula is a tautology.

Part 6: Interpretations

1. $D = \{0, 1, 2, 3, 4\}$

a	2
b	0
A	$\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle\}$
B	$\{\langle 0\rangle, \langle 2\rangle\}$
C	$\{\langle 0,0\rangle, \langle 2,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 4,0\rangle\}$

For each of the following formulas, determine if the formula is true or false on the above interpretation.

(a) $C(a, b)$: **False**

Explanation. The interpretation of a is 2 and of b is 0. The pair $\langle 2, 0\rangle$ is not in the interpretation of C.

(b) $(\forall x)(B(x) \to A(x))$: **True**

Explanation. There are two elements of the domain in the interpretation of B (0 and 2). Both of those elements are in the interpretation of A.

(c) $(\forall x)(C(x, b) \to (B(x) \lor A(x)))$: **False**

Explanation. The interpretation of b is 0. There are two elements of the domain where 0 is in the second component: $\langle 0, 0\rangle$ and $\langle 4, 0\rangle$. Element 4 is not in the interpretation of A nor in the interpretation of B.

(d) $(\exists x)(\forall y)C(x, y)$: **False**

Explanation. There is no element of the domain that is paired with every element of the domain in the interpretation of C. More formally, we are looking for an element of the domain that can be paired with every element of the domain in the interpretation of C: 0 is not the element since $\langle 0, 1\rangle$ is not in the interpretation of C, 1 is not the element, since $\langle 1, 1\rangle$ is not in the interpretation of C, 2 is not the element, since $\langle 2, 0\rangle$ is not in the interpretation of C, 3 is not the element, since $\langle 3, 1\rangle$ is not in the interpretation of C, and 4 is not the element, since $\langle 4, 2\rangle$ is not in the interpretation of C.

(e) $(\forall x)(\exists y)C(x, y)$: **True**

Explanation. Every element of the domain can be paired with some element of the domain in the interpretation of C. More formally, we have $\langle 0, 0\rangle$ in the interpretation of C, $\langle 1, 2\rangle$ in the interpretation of C, $\langle 2, 3\rangle$ in the interpretation of C, $\langle 3, 2\rangle$ in the interpretation of C, and $\langle 4, 0\rangle$ in the interpretation of C.

2. $D = \{0, 1, 2, 3, \ldots\}$ (D is the set of non-negative integers)

A	$\{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}$ (the even integers)
B	$\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle, \ldots\}$ (all integers greater than or equal to 0)
C	$\{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$ (the odd integers)

For each of the following formulas, determine if the formula is true or false on the above interpretation.

(a) $(\forall x)(A(x) \to B(x))$: **True**

Explanation. The interpretation of B is all elements of D. This is true since every even integer is an integer.

(b) $(\forall x)(C(x) \to B(x))$: **True**

Explanation. The interpretation of B is all elements of D. This is true since every odd integer is an integer.

(c) $(\forall x)(C(x) \,\&\, B(x))$: **False**

Explanation. This is false since it is not the case that every integer is odd. For instance, 2 is an element of D that is not in the interpretation of C (it is, of course, in the interpretation of D).

(d) $(\exists x)(A(x) \,\&\, C(x))$: **False**

Explanation. This is false since there is no number that is both even and odd.

(e) $(\exists x)(A(x) \to C(x))$: **True**

Explanation. This is true since there is a odd integer. For instance, 1 is an element of the interpretation of C.

3. $D = \{\mathbf{a}, \mathbf{b}\}$

a	$\mathbf{a}$
b	$\mathbf{b}$
P	$\{\langle \mathbf{a}\rangle, \langle \mathbf{b}\rangle\}$
Q	$\{\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{b}, \mathbf{a}\rangle\}$

For each of the following formulas, determine if the formula is true or false on the above interpretation.

(a) $(\forall x)Q(a,x)$: **False**

Explanation. This interpretation of a is $\mathbf{a}$. This is false since $\langle \mathbf{a}, \mathbf{a} \rangle$ is not in the interpretation of Q.

(b) $(\forall x)Q(x,b)$: **False**

Explanation. This interpretation of b is $\mathbf{b}$. This is false since $\langle \mathbf{b}, \mathbf{b} \rangle$ is not in the interpretation of Q.

(c) $(\exists x)(P(x) \,\&\, (\forall y)Q(x,y))$: **False**

Explanation. This is false since there is no element in the interpretation of P that can be paired with every element in the interpretation of Q.

(d) $(\forall x)(P(x) \,\&\, (\exists y)Q(x,y))$: **True**

Explanation. Every element of the domain is in P and can be paired with an element in interpretation of Q. More formally, $\mathbf{a}$ is in the interpretation of P and $\langle \mathbf{a}, \mathbf{b} \rangle$ is in the interpretation of Q, and $\mathbf{b}$ is in the interpretation of P and $\langle \mathbf{b}, \mathbf{a} \rangle$ is in the interpretation of Q.

(e) $(\exists y)(P(a) \,\&\, Q(y,b))$: **True**

Explanation. The interpretation of a is $\mathbf{a}$ and of b is $\mathbf{a}$. The formula is true since $\mathbf{a}$ is in the interpretation of P and $\langle \mathbf{b}, \mathbf{a} \rangle$ is in the interpretation of Q.

4. Suppose that the domain D is the set of all integers. ($D = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$)

$P(x)$	x is a positive integer
$N(x)$	x is a negative integers
$L(x,y)$	x is less than y
$G(x,y)$	x is greater than y

Translate and determine if the resulting formula is true or false in the above interpretation.

(Other answers are possible.)

(a) Every negative number is less than every negative number

$(\forall x)(N(x) \rightarrow (\forall y)(N(y) \rightarrow L(x,y)))$. This is **False**. For every negative number there is a smaller negative number.

(b) Some positive number is less than every number

$(\exists x)(P(x) \,\&\, (\forall y)L(x,y))$. This is **False**. No positive number is less than any negative number.

(c) For every positive number there is a greater positive number

$(\forall x)(\exists y)((P(x) \,\&\, P(y)) \,\&\, G(y,x))$. This is **True**.

(d) For every negative number there is a greater negative number

$(\forall x)(\exists y)(N(x) \,\&\, N(y)) \,\&\, G(y,x))$. This is **False**, since -1 is the greatest negative number (there is no negative number greater than -1).

(e) Some positive number is less than some negative number

$(\exists x)(P(x) \,\&\, (\exists y)(N(y) \,\&\, L(x,y)))$. This is **False**.

5. For each of the following formulas, if it is a contingent formula, find an interpretation that makes the formula true and one that makes it false. If the formula is contradictory or a tautology, explain why.

(a) $(\forall x)(\forall y)(P(x,y) \vee \neg P(y,x))$: **Contingent**
An interpretation that makes the formula true: The domain of interpretation is $D = \{\mathbf{a}, \mathbf{b}\}$, the interpretation of P is:

P	$\{\langle \mathbf{a}, \mathbf{b}\rangle, \langle \mathbf{b}, \mathbf{a}\rangle\}$

An interpretation that makes the formula false: The domain of interpretation is $D = \{\mathbf{a}, \mathbf{b}\}$, the interpretation of P is:

P	$\{\langle \mathbf{a}, \mathbf{b}\rangle\}$

When x is interpreted as $\mathbf{b}$ and y is interpreted as $\mathbf{a}$, since $\langle \mathbf{b}, \mathbf{a}\rangle$ is not in the interpretation of P and $\langle \mathbf{a}, \mathbf{b}\rangle$ is in the interpretation of P, we have that $P(x,y) \vee \neg P(y,x)$ is false.

(b) $\neg(\exists x)(P(x,x) \,\&\, \neg(\exists z)P(x,z))$: **Tautology**

The following truth-tree demonstrates that the formula is a tautology (all branches close).

$$\neg\neg(\exists x)(P(x,x) \,\&\, \neg(\exists z)P(x,z)) \quad \checkmark$$

$$(\exists x)(P(x,x) \,\&\, \neg(\exists z)P(x,z)) \quad \checkmark$$

$$(P(z_1,z_1) \,\&\, \neg(\exists z)P(z_1,z)) \quad \checkmark$$

$$P(z_1,z_1) \quad \checkmark$$

$$\neg(\exists z)P(z_1,z) \quad \checkmark$$

$$(\forall z)\neg P(z_1,z) \quad \checkmark$$

$$\neg P(z_1,z_1) \quad \checkmark$$

$$\text{✗}$$

(c) $(\forall x)Q(x)$ & $(\exists x)\neg Q(x)$: **Contradictory**

The following truth-tree demonstrates that the above formulas is a contradictory (all branches close).

$$(\forall x)Q(x) \mathbin{\&} (\exists x)\neg Q(x) \quad \checkmark$$

$$(\forall x)Q(z) \quad z_1$$

$$(\exists x)\neg Q(x) \quad \checkmark$$

$$\neg Q(z_1) \quad \checkmark$$

$$Q(z_1) \quad \checkmark$$

$$\text{✗}$$

(d) $(\forall x)Q(x) \vee (\forall y)\neg Q(y)$: **Contingent**.
An interpretation that makes the formula true: The domain of interpretation is $D = \{\mathbf{a}, \mathbf{b}\}$, the interpretation of Q is:

Q	$\{\langle \mathbf{a} \rangle, \langle \mathbf{b} \rangle\}$

An interpretation that makes the formula false: The domain of interpretation is $D = \{\mathbf{a}, \mathbf{b}\}$, the interpretation of Q is:

Q	$\{\langle \mathbf{a} \rangle\}$

(e) $(\forall x)(\exists y)(P(x, y) \mathbin{\&} P(y, x))$: **Contingent**.
An interpretation that makes the formula true: The domain of interpretation is $D = \{\mathbf{a}, \mathbf{b}\}$, the interpretation of P is:

P	$\{\langle \mathbf{a}, \mathbf{a} \rangle, \langle \mathbf{b}, \mathbf{b} \rangle\}$

An interpretation that makes the formula false: The domain of interpretation is $D = \{\mathbf{a}, \mathbf{b}\}$, the interpretation of P is:

P	$\{\langle \mathbf{a}, \mathbf{a} \rangle\}$

Part 7: Deductions

Complete the following deductions.

1.

1.	$(\forall x)(F(x) \to G(x))$	Premise
2.	$\neg(F(c) \,\&\, G(c))$	Premise
3.	$F(c)$	Assumption
4.	$F(c) \to G(c)$	$\forall$E : 1
5.	$G(c)$	$\to$E : 3, 4
6.	$F(c) \,\&\, G(c)$	&I : 3, 5
7.	$\bot$	$\bot$I : 2, 6
8.	$\neg F(c)$	$\neg$I : 7

2.

1.	$(\forall x)((F(x) \lor G(x)) \to H(x))$	Premise
2.	$(\forall x)(H(x) \to (J(x) \,\&\, K(x)))$	Premise
3.	$(F(a) \lor G(a)) \to H(a)$	$\forall$E : 1
4.	$H(a) \to (J(a) \,\&\, K(a))$	$\forall$E : 2
5.	$F(a)$	Assumption
6.	$F(a) \lor G(a)$	$\lor$IL: 5
7.	$H(a)$	$\to$E : 3, 6
8.	$J(a) \,\&\, K(a)$	$\to$E : 7, 4
9.	$K(a)$	&ER : 8
10.	$F(a) \to K(a)$	$\to$I : 9

3.

1.	$(\forall x)((F(x) \,\&\, G(x)) \to H(x))$	Premise
2.	$(F(a) \,\&\, \neg H(a))$	Premise
3.	$(F(a) \,\&\, G(a)) \to H(a)$	$\forall$E : 1
4.	$G(a)$	Assumption
5.	$F(a)$	&EL : 5
6.	$F(a) \,\&\, G(a)$	&I : 5, 4
7.	$H(a)$	$\to$E : 6, 3
8.	$\neg H(a)$	&ER : 2
9.	$\bot$	$\bot$I : 7, 8
10.	$\neg G(a)$	$\neg$I : 9

4.

1.	$(\forall x)(F(x) \to R(x,x))$	Premise
2.	$(\forall x)\neg R(a,x)$	Premise
3.	$F(a) \to R(a,a)$	$\forall$E: 1
4.	$F(a)$	Assumption
5.	$R(a,a)$	$\to$E: 4, 3
6.	$\neg R(a,a)$	$\forall$E: 2
7.	$\bot$	$\bot$I: 5, 6
8.	$\neg F(a)$	$\neg$I: 7

5.

1.	$(\forall x)(F(x) \to G(x))$	Premise
2.	$F(a)$	Premise
3.	$F(a) \to G(a)$	$\forall$E: 1
4.	$G(a)$	$\to$E: 4, 3
5.	$(\exists x)G(x)$	$\exists$I: 5

6.

1.	$\neg(\exists x)(F(x) \,\&\, G(x))$	Premise
2.	$F(a)$	Premise
3.	$G(a)$	Assumption
4.	$F(a) \,\&\, G(a)$	&I: 2, 3
5.	$(\exists x)(F(x) \,\&\, G(x))$	$\exists$I: 4
6.	$\bot$	$\bot$I: 1, 5
7.	$\neg G(a)$	$\neg$I: 6

7.

1.	$(\exists x)F(x) \to (\forall x)G(x)$	Premise
2.	$F(a)$	Premise
3.	$(\exists x)F(x)$	$\exists$I: 2
4.	$(\forall x)G(x)$	$\to$E: 3, 1
5.	$G(a)$	$\forall$E: 4

8.

1.	$(\forall x)((F(x) \vee G(x)) \rightarrow H(x))$	Premise
2.	$\neg(G(a) \vee H(a))$	Premise
3.	$(F(a) \vee G(a)) \rightarrow H(a)$	$\forall$E : 1
4.	$F(a)$	Assumption
5.	$F(a) \vee G(a)$	$\vee$IR : 4
6.	$H(a)$	$\rightarrow$E : 5, 3
7.	$G(a) \vee H(a)$	$\vee$IL: 6
8.	$\bot$	$\bot$I : 2, 7
9.	$\neg F(a)$	$\neg$I : 8
10.	$(\exists x)\neg F(x)$	$\exists$I : 9

9.

1.	$(\forall x)(F(x) \rightarrow G(x))$	Premise
2.	$(\exists x)(F(x)\ \&\ H(x))$	Premise
3.	$F(z_1)\ \&\ H(z_1)$	Assumption
4.	$F(z_1) \rightarrow G(z_1)$	$\forall$E : 1
5.	$F(z_1)$	&EL : 3
6.	$G(z_1)$	$\rightarrow$E : 5, 4
7.	$H(z_1)$	&ER : 3
8.	$G(z_1)\ \&\ H(z_1)$	&I : 6, 7
9.	$(\exists x)(G(x)\ \&\ H(x))$	$\exists$I : 8
10.	$(\exists x)(G(x)\ \&\ H(x))$	$\exists$E : 2, 9

10.

1.	$(\forall x)(F(x) \rightarrow \neg G(x))$	Premise
2.	$(\exists x)(F(x)\ \&\ G(x))$	Assumption
3.	$F(z_1)\ \&\ G(z_1)$	Assumption
4.	$F(z_1) \rightarrow \neg G(z_1)$	$\forall$E : 1
5.	$F(z_1)$	&EL : 3
6.	$\neg G(z_1)$	$\rightarrow$E : 5, 4
7.	$G(z_1)$	&ER : 3
8.	$\bot$	$\bot$I : 6, 7
9.	$\bot$	$\exists$E : 2, 8
10.	$\neg(\exists x)(F(x)\ \&\ G(x))$	$\neg$I : 9

11.

1.	$(\exists x)(F(x) \mathbin{\&} \neg G(x))$	Premise
2.	$(\forall x)(F(x) \to G(x))$	Assumption
3.	$F(z_1) \mathbin{\&} \neg G(z_1)$	Assumption
4.	$F(z_1) \to G(z_1)$	$\forall$E : 2
5.	$F(z_1)$	&EL : 3
6.	$G(z_1)$	$\to$E : 5, 4
7.	$\neg G(z_1)$	&ER : 3
8.	$\bot$	$\bot$I : 6, 7
9.	$\bot$	$\exists$E : 1, 8
10.	$\neg(\forall x)(F(x) \to G(x))$	$\neg$I : 9

12.

1.	$(\forall x)(F(x) \to G(x))$	Premise
2.	$(\forall x)(G(x) \to \neg H(x))$	Premise
3.	$(\exists x)(F(x) \mathbin{\&} H(x))$	Assumption
4.	$F(z_1) \mathbin{\&} H(z_1)$	Assumption
5.	$F(z_1)$	&EL : 4
6.	$F(z_1) \to G(z_1)$	$\forall$E : 1
7.	$G(z_1)$	$\to$E : 5, 6
8.	$G(z_1) \to \neg H(z_1)$	$\forall$E : 2
9.	$\neg H(z_1)$	$\to$E : 7, 8
10.	$H(z_1)$	&ER : 4
11.	$\bot$	$\bot$I : 9, 10
12.	$\bot$	$\exists$E : 3, 11
13.	$\neg(\exists x)(F(x) \mathbin{\&} H(x))$	$\neg$I : 12

13.

1.	$(\forall x)(G(x) \to H(x))$	Premise
2.	$(\exists x)(I(x) \mathbin{\&} \neg H(x))$	Premise
3.	$(\forall x)(\neg F(x) \lor G(x))$	Premise
4.	$I(z_1) \mathbin{\&} \neg H(z_1)$	Assumption
5.	$\neg H(z_1)$	&ER : 4
6.	$G(z_1) \to H(z_1)$	∀E : 1
7.	$G(z_1)$	Assumption
8.	$H(z_1)$	→E : 7, 6
9.	$\bot$	⊥I : 5, 8
10.	$\neg G(z_1)$	¬I : 9
11.	$\neg F(z_1) \lor G(z_1)$	∀E : 3
12.	$\neg F(z_1)$	Assumption
13.	$G(z_1)$	Assumption
14.	$\bot$	⊥I : 10, 13
15.	$\neg F(z_1)$	⊥E : 14
16.	$\neg F(z_1)$	∨E : 11, 12, 15
17.	$I(z_1)$	&EL : 4
18.	$I(z_1) \mathbin{\&} \neg F(z_1)$	&I : 17, 16
19.	$(\exists x)(I(x) \mathbin{\&} \neg F(x))$	∃I : 18
20.	$(\exists x)(I(x) \mathbin{\&} \neg F(x))$	∃E : 19, 2

14.

1.	$(\exists x)F(x) \rightarrow (\forall x)\neg G(x)$	Premise
2.	$F(w)$	Assumption
3.	$(\exists x)F(x)$	$\exists$I : 2
4.	$(\forall x)\neg G(x)$	$\rightarrow$E : 3, 1
5.	$(\exists y)G(y)$	Assumption
6.	$G(z_1)$	Assumption
7.	$\neg G(z_1)$	$\forall$E : 4
8.	$\bot$	$\bot$I : 6, 7
9.	$\bot$	$\exists$E : 5, 8
10.	$\neg(\exists y)G(y)$	$\neg$E : 9
11.	$F(w) \rightarrow \neg(\exists y)G(y)$	$\rightarrow$I : 10
12.	$(\forall x)(F(x) \rightarrow \neg(\exists y)G(y))$	$\forall$I : 11

15.

1.	$(\exists x)\neg(\exists y)(F(y) \mathbin{\&} R(x,y))$	Premise
2.	$\neg(\exists y)(F(y) \mathbin{\&} R(z_1,y))$	Assumption
3.	$F(w)$	Assumption
4.	$R(z_1,w)$	Assumption
5.	$F(w) \mathbin{\&} R(z_1,w)$	&I : 3, 4
6.	$(\exists y)(F(y) \mathbin{\&} R(z_1,y))$	$\exists$I : 5
7.	$\bot$	$\bot$I : 2, 6
8.	$\neg R(z_1,y)$	$\neg$I : 7
9.	$(\exists y)\neg R(y,w)$	$\exists$I : 8
10.	$(F(w) \rightarrow (\exists y)\neg R(y,w))$	$\rightarrow$I : 9
11.	$(\forall x)(F(x) \rightarrow (\exists y)\neg R(y,x))$	$\forall$I : 10
12.	$(\forall x)(F(x) \rightarrow (\exists y)\neg R(y,x))$	$\exists$E : 1, 11

16.

1.	$(\forall x)(F(x) \to (\exists y)\neg K(x,y))$	Premise
2.	$(\exists x)(G(x) \mathbin{\&} (\forall y)K(x,y))$	Premise
3.	$G(z_1) \mathbin{\&} (\forall y)K(z_1,y)$	Assumption
4.	$F(z_1) \to (\exists y)\neg K(z_1,y)$	$\forall$E : 1
5.	$F(z_1)$	Assumption
6.	$(\exists y)\neg K(z_1,y)$	$\to$E : 5, 4
7.	$\neg K(z_1,z_2)$	Assumption
8.	$(\forall y)K(z_1,y)$	&ER : 2
9.	$K(z_1,z_2)$	$\forall$E : 8
10.	$\bot$	$\bot$I : 7, 9
11.	$\bot$	$\exists$E : 6, 10
12.	$\neg F(z_1)$	$\neg$I : 11
13.	$G(z_1)$	&EL : 3
14.	$G(z_1) \mathbin{\&} \neg F(z_1)$	&I : 13, 12
15.	$(\exists x)(G(x) \mathbin{\&} \neg F(x))$	$\exists$I : 14
16.	$(\exists x)(G(x) \mathbin{\&} \neg F(x))$	$\exists$E : 2, 15

17.

1.	$(\forall x)(\exists y)R(x,y)$	Premise
2.	$(\forall x)((\exists y)R(x,y) \to R(x,x))$	Premise
3.	$(\forall x)(R(x,x) \to (\forall y)R(y,x))$	Premise
4.	$(\exists y)R(v,y)$	$\forall$E$: 1$
5.	$(\exists y)R(v,y) \to R(v,v)$	$\forall$E$: 2$
6.	$R(v,v)$	$\to$E$: 4, 5$
7.	$R(v,v) \to (\forall y)R(y,v)$	$\forall$E$: 3$
8.	$(\forall y)R(y,v)$	$\to$E$: 6, 7$
9.	$R(w,v)$	$\forall$E$: 8$
10.	$(\forall y)R(w,y)$	$\forall$I$: 9$
11.	$(\forall x)(\forall y)R(x,y)$	$\forall$I$: 10$

18.

1.	$(\exists x)(\exists y)R(x,y)$	Premise
2.	$(\forall x)((\exists y)R(x,y) \to (\forall y)R(y,x))$	Premise
3.	$(\exists y)R(z_1,y)$	Assumption
4.	$(\exists y)R(z_1,y) \to (\forall y)R(y,z_1)$	$\forall$E: 2
5.	$(\forall y)R(y,z_1)$	$\to$E: 3, 4
6.	$R(v,z_1)$	$\forall$E: 5
7.	$(\exists y)R(v,y)$	$\exists$I: 6
8.	$(\exists y)R(v,y) \to (\forall y)R(y,v)$	$\forall$E: 2
9.	$(\forall y)R(y,v)$	$\to$E: 7, 8
10.	$R(w,v)$	$\forall$E: 9
11.	$R(w,v)$	$\exists$E: 1, 10
12.	$(\forall y)R(w,y)$	$\forall$I: 11
13.	$(\forall x)(\forall y)R(x,y)$	$\forall$I: 12

19.

1.	$(\exists x)F(x) \to (\forall x)F(x)$	Premise
2.	$F(w)$	Assumption
3.	$(\exists x)F(x))$	$\exists$I$: 2$
4.	$(\forall x)F(x))$	$\to$E$: 3, 1$
5.	$F(v)$	$\forall$E$: 4$
6.	$F(v)$	Assumption
7.	$(\exists x)F(x))$	$\exists$I$: 6$
8.	$(\forall x)F(x))$	$\to$E$: 7, 1$
9.	$F(w)$	$\forall$E$: 8$
10.	$F(w) \leftrightarrow F(v)$	$\leftrightarrow$I $: 5, 9$
11.	$(\forall y)(F(w) \leftrightarrow F(y))$	$\forall$I$: 10$
12.	$(\forall x)(\forall y)(F(x) \leftrightarrow F(y))$	$\forall$I$: 11$

20.

1.	$(\forall x)(F(x) \to G(x))$	Premise
2.	$\neg(\exists x)(G(x) \mathbin{\&} H(x))$	Premise
3.	$(\exists x)(F(x) \mathbin{\&} H(x))$	Assumption
4.	$F(z_1) \mathbin{\&} H(z_1)$	Assumption
5.	$F(z_1) \to G(z_1)$	$\forall$E : 1
6.	$F(z_1)$	&EL : 4
7.	$G(z_1)$	$\to$E : 6, 5
8.	$H(z_1)$	&ER : 4
9.	$G(z_1) \mathbin{\&} H(z_1)$	&I : 7, 8
10.	$(\exists x)(G(x) \mathbin{\&} H(x))$	$\exists$I : 9
11.	$\bot$	$\bot$I : 2, 10
12.	$\bot$	$\exists$E : 3, 11
13.	$\neg(\exists x)(F(x) \mathbin{\&} H(x))$	$\neg$I : 12