Introduction to Logic

PHIL 170

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Announcements

- read Chapter 11 (and Chapter 12).
- Do the practice problems!
- Schedule
  1. Mon., 11/30: Ch. 11 (Deductions for Predicate Logic)
  2. Wed., 12/2: Ch. 11, course summary
  3. Mon., 12/7: Concluding remarks (general perspectives)
  4. Wed., 12/9: Concluding remarks (general perspectives)
- **Midterm #3**: Quiz (12/1, 11.59pm), Lab - Truth-Trees and Lab - Derivations (12/2, 11.59pm)
- In-class quiz in sections: A deduction similar to the practice problems in Chapter 11.
- **Final Exam**: Wed., Dec 16, 8:00am - 10:00am, LEF 2205

Final Exam

- Basic concepts: Logical connectives, tautology, contradictory, contingent formula, valid/invalid arguments, truth-value function, interpretations
- Translations into predicate logic, determine if a formula is true in a model.
- Determine if a pair of formulas are logically equivalent.
- Determine if an argument is valid or invalid. Find a counterexample if the argument is not valid.
  - Using truth-tables (you need to know the truth-table rules)
  - Using truth-trees (the decompositions rules will be provided)
- Find deductions in propositional logic/predicate logic (deduction rules will be provided).

- **Tautology**: A formula of predicate logic is a tautology just in case it is true on every interpretation.
- **Contradictory Formula**: A formula of predicate logic is a contradictory just in case it is false on every interpretation.
- **Contingent Formula**: A formula of predicate logic is contingent just in case it is true on some interpretations, and false on others.

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Example

Suppose that the domain D is the set of integers.

$P(x)$	x is a positive integer
$N(x)$	x is a negative integers
$G(x, y)$	x is greater than y

Translate and determine if the resulting formula is true or false in the above interpretation.

1. Every positive number is greater than every negative number
2. Some positive number is greater than every number
3. For every positive number there is a greater positive number
4. For every negative number there is a greater negative number
5. Some positive number is greater than some negative number

Example

Suppose that the domain D is the set of integers.

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Translate and determine if the resulting formula is true or false in the above interpretation.

1. Every positive number is greater than every negative number

$(\forall x)(P(x) \to (\forall y)(N(y) \to G(x, y))$. True

Example

Suppose that the domain D is the set of integers.

$P(x)$	x is a positive integer
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$G(x, y)$	x is greater than y

Translate and determine if the resulting formula is true or false in the above interpretation.

2. Some positive number is greater than every number

   $(\exists x)(P(x) \;\&\; (\forall y)G(x, y))$ False

Example

Suppose that the domain D is the set of integers.

$P(x)$	x is a positive integer
$N(x)$	x is a negative integers
$G(x, y)$	x is greater than y

Translate and determine if the resulting formula is true or false in the above interpretation.

3. For every positive number there is a greater positive number

   $(\forall x)(\exists y)((P(x) \,\&\, P(y)) \rightarrow G(y, x))$ True

Example

Suppose that the domain D is the set of integers.

$P(x)$	x is a positive integer
$N(x)$	x is a negative integers
$G(x, y)$	x is greater than y

Translate and determine if the resulting formula is true or false in the above interpretation.

4. For every negative number there is a greater negative number

$(\forall x)(N(x) \rightarrow (\exists y)(N(y)\ \&\ G(y, x)))$ False

Example

Suppose that the domain D is the set of integers.

$P(x)$	x is a positive integer
$N(x)$	x is a negative integers
$G(x, y)$	x is greater than y

Translate and determine if the resulting formula is true or false in the above interpretation.

5. Some positive number is greater than some negative number

   $(\exists x)(P(x) \; \& \; (\exists y)(N(y) \; \& \; G(x, y))$ True

Deduction for Predicate Logic

Conjunction Introduction (&I)

$p1.$	φ	
$p2.$	ψ	
	$\vdots$	
c.	$(\varphi \mathbin{\&} \psi)$	&I: $p1, p2$

Conjunction Elimination (&EL, &ER)

$p1.$	$(\varphi \mathbin{\&} \psi)$	
	$\vdots$	
c.	φ	&EL : $p1$

$p1.$	$(\varphi \mathbin{\&} \psi)$	
	$\vdots$	
c.	ψ	&ER : $p1$

Conditional Introduction ($\rightarrow$ I)

*a*1.	φ	Assumption
	$\vdots$	
*p*1.	ψ	Goal
c.	$(\varphi \rightarrow \psi)$	$\rightarrow$I : *p*1

*p*1.	ψ	
	$\vdots$	
c.	$(\varphi \rightarrow \psi)$	$\rightarrow$I : *p*1

Conditional Elimination ($\rightarrow$ E)

$p1.$	φ	
$p2.$	$(\varphi \rightarrow \psi)$	
	$\vdots$	
c.	ψ	$\rightarrow$E : $p1$, $p2$

Disjunction Introduction ($\vee$IL, $\vee$IR)

$p1.$	φ	
	$\vdots$	
c.	$(\varphi \vee \psi)$	$\vee$IR : $p1$

$p1.$	φ	
	$\vdots$	
c.	$(\psi \vee \varphi)$	$\vee$IL: $p1$

Disjunction Elimination ($\vee$E)

$p1.$	$(\varphi \vee \psi)$	Premise
$a1.$	φ	Assumption
	$\vdots$	
$p2.$	ρ	Goal
$a2.$	ψ	Assumption
	$\vdots$	
$p3.$	ρ	Goal
$c.$	ρ	$\vee$E: $p1, p2, p3$

Negation Introduction/Elimination ($\neg$I, $\neg$E)

$a1.$	φ	Assumption
	$\vdots$	
$p1.$	$\bot$	Goal
c.	$\neg\varphi$	$\neg$I : $p1$

$a1.$	$\neg\varphi$	Assumption
	$\vdots$	
$p1.$	$\bot$	Goal
c.	φ	$\neg$E : $p1$

Falsum Introduction/Elimination ($\perp$I, $\perp$E)

$p1.$	φ	
$p2.$	$\neg\varphi$	
	$\vdots$	
c.	$\perp$	$\perp$I : $p1, p2$

$p1.$	$\perp$	
	$\vdots$	
c.	φ	$\perp$E : $p1$

Biconditional Introduction ($\leftrightarrow$I)

*a*1.	φ	Assumption
	$\vdots$	
*p*1.	ψ	Goal
*a*2.	ψ	Assumption
	$\vdots$	
*p*2.	φ	Goal
c.	$(\varphi \leftrightarrow \psi)$	$\leftrightarrow$I: *p*1, *p*2

*p*1.	φ	
*p*2.	ψ	
	$\vdots$	
c.	$(\varphi \leftrightarrow \psi)$	$\leftrightarrow$I: *p*1, *p*2

Biconditional Elimination ($\leftrightarrow$E)

*p*1.	$(\varphi \leftrightarrow \psi)$	
*p*2.	φ	
	$\vdots$	
c.	ψ	$\leftrightarrow$E : *p*1, *p*2

*p*1.	$(\varphi \leftrightarrow \psi)$	
*p*2.	ψ	
	$\vdots$	
c.	φ	$\leftrightarrow$E : *p*1, *p*2

Universal Elimination ($\forall$E)

$p1.$	$(\forall x)\varphi$	
	$\vdots$	
c.	$\varphi[\tau/x]$	$\forall$E : $p1$

Universal Elimination ($\forall$E), Examples

1.	$(\forall x)(\forall y)R(x,y)$	Premise
2.	$(\forall u)(\forall v)Q(v,u)$	Premise
	$\vdots$	
n.	$R(a,b)$ & $Q(t,s)$	Goal

Universal Elimination ($\forall$E), Examples

1.	$(\forall x)(\forall y)R(x, y)$	Premise
2.	$(\forall u)(\forall v)Q(v, u)$	Premise
	$\vdots$	
n.	$R(a, b)$ & $Q(t, s)$	Goal

1.	$(\forall x)(P(x) \rightarrow Q(x, x))$	Premise
2.	$\neg Q(b, b)$	Premise
	$\vdots$	
n.	$\neg(\forall x)P(x)$	Goal

Existential Introduction (∃I)

$p1.$	$\varphi[\tau/x]$	
	$\vdots$	
c.	$(\exists x)\varphi$	$\exists\mathsf{I} : p1$

1.	$(\forall x)P(x)$	Premise
	$\vdots$	
n.	$(\exists x)P(x)$	Goal

1.	$(\forall z)R(z,z)$	Premise
2.	$(\forall y)\neg O(y,y)$	Premise
	$\vdots$	
n.	$(\exists x)(\exists y)(R(x,y)\ \&\ \neg O(y,x))$	Goal

$p1.$	$(\exists u)\varphi$		
$a1.$	$\varphi[v/u]$		Assumption
	$\vdots$		
$p2.$	ψ		Goal
c.	ψ		$\exists$E : $p1$, $p2$

1. v is a variable,
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)P(x)$	Premise
2.	$(\forall x)(P(x) \to Q(x))$	Premise
	$\vdots$	
n.	$(\exists x)Q(x)$	Goal

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4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$P(a) \to Q(a)$	Premise
2.	$(\exists x)P(x)$	Premise
3.	$P(a)$	Assumption
4.	$Q(a)$	$\to$E : 1, 3
5.	$Q(a)$	$\exists$E : 2, 4, *BAD*!

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4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)(\forall y)R(x,y)$	Premise
2.	$(\forall y)R(y,y)$	Assumption
3.	$R(a,a)$	$\forall$E : 1, 3
4.	$R(a,a)$	$\exists$E : 1, 3, *BAD*!

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2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)(\forall y)R(x,y)$	Premise
2.	$(\forall y)R(y,y)$	Assumption
3.	$R(a,a)$	$\forall$E : 1, 3
4.	$R(a,a)$	$\exists$E : 1, 3, *BAD*!

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1.	$(\exists x)P(x)$	Premise
2.	$(\exists x)Q(x)$	Premise
3.	$P(v)$	Assumption
4.	$Q(v)$	Assumption
5.	$P(v)\ \&\ Q(v)$	$\&I: 3, 4$
6.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists I: 5$
7.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists E: 2, 6$ *BAD!*
8.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists E: 1, 7$

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2.	$(\exists x)Q(x)$	Premise
3.	$P(v)$	Assumption
4.	$Q(v)$	Assumption
5.	$P(v)\ \&\ Q(v)$	$\&\mathrm{I}: 3, 4$
6.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists\mathrm{I}: 5$
7.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists\mathrm{E}: 2, 6$ *BAD!*
8.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists\mathrm{E}: 1, 7$

Universal Introduction ($\forall$I)

$p1.$	$\varphi[v/u]$	
	$\vdots$	
c.	$(\forall u)\varphi$	$\forall$I : $p1$

1. v is a variable,
2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)A(x)$	Premise
2.	$(\forall x)B(x)$	Premise
	$\vdots$	
n.	$(\forall x)(A(x) \,\&\, B(x))$	Goal

1. v is a variable,
2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)P(x,x)$	Premise
2.	$P(w,w)$	$\forall$E : 1
3.	$(\forall x)P(x,w)$	&I : 2*BAD*!

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2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)P(x,x)$	Premise
2.	$P(w,w)$	$\forall$E : 1
3.	$(\forall x)P(x,w)$	&I : 2BAD!
4.	$(\forall y)(\forall x)P(x,y)$	&I : 3BAD!

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2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)P(x,x)$	Premise
2.	$P(w,w)$	$\forall$E : 1
3.	$(\forall x)P(x,x)$	&I : 2

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3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)(\forall y)P(x, y)$	Premise
2.	$(\forall y)P(w, y)$	$\forall$E : 1
3.	$P(w, w)$	$\forall$E : 2
4.	$(\forall x)P(x, x)$	$\forall$I : 3

Truth-trees for predicate logic

$$
\begin{array}{c}
(\varphi \,\&\, \psi) \\
\varphi \\
\psi
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & \neg(\varphi \,\&\, \psi) & \\
\diagup & & \diagdown \\
\neg\varphi & & \neg\psi
\end{array}
$$

$$
\begin{array}{ccc}
 & (\varphi \vee \psi) & \\
\diagup & & \diagdown \\
\varphi & & \psi
\end{array}
\qquad\qquad
\begin{array}{c}
\neg(\varphi \vee \psi) \\
\neg\varphi \\
\neg\psi
\end{array}
$$

$$\begin{array}{cc} \multicolumn{2}{c}{(\varphi \to \psi)} \\ / & \backslash \\ \neg\varphi & \psi \end{array}$$

$$\begin{array}{c} \neg(\varphi \to \psi) \\ \varphi \\ \neg\psi \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{(\varphi \leftrightarrow \psi)} \\ / & \backslash \\ \varphi & \neg\varphi \\ \psi & \neg\psi \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{\neg(\varphi \leftrightarrow \psi)} \\ / & \backslash \\ \varphi & \psi \\ \neg\psi & \neg\varphi \end{array}$$

$$\begin{array}{c} \neg\neg\varphi \\ \varphi \end{array}$$

Decomposition Rules for Quantifiers

$$\frac{(\exists u)\varphi}{\varphi[v/u]}$$

Provided v does not appear on the branch

$$\frac{(\forall u)\varphi}{\varphi[t/u]}$$

$$\frac{\neg(\exists u)\varphi}{(\forall u)\neg\varphi}$$

$$\frac{\neg(\forall u)\varphi}{(\exists u)\neg\varphi}$$

When is a branch completed?

For every decomposition rule, except the universal decomposition rule, when it is applied, check off the formula. *Universally quantified formulas are never checked off*.

admissible term: any constant or variable that has a **free occurrence** in a formula on the branch.

A truth-tree is **completed** once any formula on an open branch is either an atomic formula, the negation of an atomic formula, checked off, or a universally quantified formula that has been instantiated with every admissible term.