Introduction to Logic

PHIL 170

Eric Pacuit

University of Maryland, College Park
pacuit.org
epacuit@umd.edu

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Announcements

- read Chapter 11 (and Chapter 12).
- Do the practice problems!
- Schedule
  1. Mon., 11/23: Ch. 11 (Deductions for Predicate Logic)
  2. Wed., 11/25: Ch. 11 (Deductions for Predicate Logic)
  3. Mon., 11/30: Ch. 11 (Deductions for Predicate Logic)
  4. Wed., 12/2: Ch. 11, course summary
  5. Mon., 12/7: Concluding remarks (general perspectives)
  6. Wed., 12/9: Concluding remarks (general perspectives)
- **Midterm #3**: Quiz (12/1, 11.59pm), Lab - Truth-Trees and Lab - Derivations (12/2, 11.59pm)
- **Final Exam**: Wed., Dec 16, 8:00am - 10:00am, LEF 2205

Final Exam

- Basic concepts: Logical connectives, tautology, contradictory, contingent formula, valid/invalid arguments, truth-value function, interpretations
- Translations into predicate logic
- Determine if a pair of formulas are logically equivalent.
- Determine if an argument is valid or invalid. Find a counterexample if the argument is not valid.
  - Using truth-tables (you need to know the truth-table rules)
  - Using truth-trees (the decompositions rules will be provided)
- Find deductions in propositional logic/predicate logic (deduction rules will be provided).

- **Tautology**: A formula of predicate logic is a tautology just in case it is true on every interpretation.
- **Contradictory Formula**: A formula of predicate logic is a contradictory just in case it is false on every interpretation.
- **Contingent Formula**: A formula of predicate logic is contingent just in case it is true on some interpretations, and false on others.

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Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u.

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- $((\forall x)P(x) \;\&\; R(x,y))[u/x] = ((\forall x)P(x) \;\&\; R(u,y))$

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Truth-trees for predicate logic

$$(\varphi \mathbin{\&} \psi)$$
$$\varphi$$
$$\psi$$

$$\neg(\varphi \mathbin{\&} \psi)$$
$$\neg\varphi \qquad\qquad \neg\psi$$

$$(\varphi \vee \psi)$$
$$\varphi \qquad\qquad \psi$$

$$\neg(\varphi \vee \psi)$$
$$\neg\varphi$$
$$\neg\psi$$

$$\begin{array}{cc} \multicolumn{2}{c}{(\varphi \to \psi)} \\ \neg\varphi & \psi \end{array}$$

$$\begin{array}{c} \neg(\varphi \to \psi) \\ \varphi \\ \neg\psi \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{(\varphi \leftrightarrow \psi)} \\ \varphi & \neg\varphi \\ \psi & \neg\psi \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{\neg(\varphi \leftrightarrow \psi)} \\ \varphi & \psi \\ \neg\psi & \neg\varphi \end{array}$$

$$\begin{array}{c} \neg\neg\varphi \\ \varphi \end{array}$$

Decomposition Rules for Quantifiers

$$\frac{(\exists u)\varphi}{\varphi[v/u]}$$

Provided v does not appear on the branch

$$\frac{(\forall u)\varphi}{\varphi[t/u]}$$

$$\frac{\neg(\exists u)\varphi}{(\forall u)\neg\varphi}$$

$$\frac{\neg(\forall u)\varphi}{(\exists u)\neg\varphi}$$

When is a branch completed?

For every decomposition rule, except the universal decomposition rule, when it is applied, check off the formula. *Universally quantified formulas are never checked off*.

admissible term: any constant or variable that has a **free occurrence** in a formula on the branch.

A truth-tree is **completed** once any formula on an open branch is either an atomic formula, the negation of an atomic formula, checked off, or a universally quantified formula that has been instantiated with every admissible term.

Which arguments are quantificationally valid?

1.
$$\begin{array}{l} (\forall x)(H(x) \rightarrow M(x)) \\ H(s) \\ \hline \therefore M(s) \end{array}$$

2.
$$\begin{array}{l} (\forall x)(H(x) \rightarrow M(x)) \\ \neg H(a) \\ \hline \therefore \neg M(a) \end{array}$$

3.
$$\begin{array}{l} (\exists x)(S(x) \;\&\; C(x)) \\ (\exists x)(S(x) \;\&\; D(x)) \\ \hline \therefore (\exists x)(S(x) \;\&\; (D(x) \;\&\; C(x))) \end{array}$$

4.
$$\begin{array}{l} (\forall x)(P(x) \rightarrow Q(x)) \\ (\forall x)(Q(x) \rightarrow R(x)) \\ \hline \therefore (\forall x)(P(x) \rightarrow R(x)) \end{array}$$

Deduction for Predicate Logic

Conjunction Introduction (&I)

$p1$.	φ	
$p2$.	ψ	
	$\vdots$	
c.	$(\varphi \,\&\, \psi)$	&I: $p1, p2$

Conjunction Elimination (&EL, &ER)

$p1.$	$(\varphi \,\&\, \psi)$	
	$\vdots$	
c.	φ	&EL : $p1$

$p1.$	$(\varphi \,\&\, \psi)$	
	$\vdots$	
c.	ψ	&ER : $p1$

Conditional Introduction ($\rightarrow$ I)

$a1.$	φ	Assumption
	$\vdots$	
$p1.$	ψ	Goal
c.	$(\varphi \rightarrow \psi)$	$\rightarrow$I : $p1$

$p1.$	ψ	
	$\vdots$	
c.	$(\varphi \rightarrow \psi)$	$\rightarrow$I : $p1$

Conditional Elimination ($\rightarrow$ E)

$p1.$	φ	
$p2.$	$(\varphi \rightarrow \psi)$	
	$\vdots$	
c.	ψ	$\rightarrow$E : $p1$, $p2$

Disjunction Introduction ($\vee$IL, $\vee$IR)

$p1.$	φ	
	$\vdots$	
c.	$(\varphi \vee \psi)$	$\vee$IR : $p1$

$p1.$	φ	
	$\vdots$	
c.	$(\psi \vee \varphi)$	$\vee$IL: $p1$

Disjunction Elimination ($\vee$E)

*p*1.	$(\varphi \vee \psi)$	Premise
*a*1.	φ	Assumption
	$\vdots$	
*p*2.	ρ	Goal
*a*2.	ψ	Assumption
	$\vdots$	
*p*3.	ρ	Goal
c.	ρ	$\vee$E: *p*1, *p*2, *p*3

Negation Introduction/Elimination ($\neg$I, $\neg$E)

$a1.$	φ	Assumption
	$\vdots$	
$p1.$	$\bot$	Goal
c.	$\neg\varphi$	$\neg$I : $p1$

$a1.$	$\neg\varphi$	Assumption
	$\vdots$	
$p1.$	$\bot$	Goal
c.	φ	$\neg$E : $p1$

Falsum Introduction/Elimination ($\bot$I, $\bot$E)

$p1.$	φ	
$p2.$	$\neg\varphi$	
	$\vdots$	
c.	$\bot$	$\bot$I : $p1, p2$

$p1.$	$\bot$	
	$\vdots$	
c.	φ	$\bot$E : $p1$

Biconditional Introduction ($\leftrightarrow$I)

$a1.$	φ	Assumption
	$\vdots$	
$p1.$	ψ	Goal
$a2.$	ψ	Assumption
	$\vdots$	
$p2.$	φ	Goal
$c.$	$(\varphi \leftrightarrow \psi)$	$\leftrightarrow$I: $p1, p2$

$p1.$	φ	
$p2.$	ψ	
	$\vdots$	
c.	$(\varphi \leftrightarrow \psi)$	$\leftrightarrow$I: $p1, p2$

Biconditional Elimination ($\leftrightarrow$E)

$p1.$	$(\varphi \leftrightarrow \psi)$	
$p2.$	φ	
	$\vdots$	
c.	ψ	$\leftrightarrow$E : $p1, p2$

$p1.$	$(\varphi \leftrightarrow \psi)$	
$p2.$	ψ	
	$\vdots$	
c.	φ	$\leftrightarrow$E : $p1, p2$

Universal Elimination (∀E)

$p1.$	$(\forall x)\varphi$	
	$\vdots$	
c.	$\varphi[\tau/x]$	$\forall\mathrm{E}: p1$

Universal Elimination ($\forall$E), Examples

1.	$(\forall x)(\forall y)R(x,y)$	Premise
2.	$(\forall u)(\forall v)Q(v,u)$	Premise
	$\vdots$	
n.	$R(a,b)\ \&\ Q(t,s)$	Goal

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1.	$(\forall x)(\forall y)R(x,y)$	Premise
2.	$(\forall u)(\forall v)Q(v,u)$	Premise
	$\vdots$	
$n.$	$R(a,b)\ \&\ Q(t,s)$	Goal

1.	$(\forall x)(P(x) \to Q(x,x))$	Premise
2.	$\neg Q(b,b)$	Premise
	$\vdots$	
$n.$	$\neg(\forall x)P(x)$	Goal

Existential Introduction ($\exists$I)

$p1.$	$\varphi[\tau/x]$	
	$\vdots$	
c.	$(\exists x)\varphi$	$\exists$I : $p1$

1.	$(\forall x)P(x)$	Premise
	$\vdots$	
n.	$(\exists x)P(x)$	Goal

1.	$(\forall z)R(z,z)$	Premise
2.	$(\forall y)\neg O(y,y)$	Premise
	$\vdots$	
n.	$(\exists x)(\exists y)(R(x,y)\ \&\ \neg O(y,x))$	Goal

$p1.$	$(\exists u)\varphi$	
$a1.$	$\varphi[v/u]$	Assumption
	$\vdots$	
$p2.$	ψ	Goal
c.	ψ	$\exists$E : $p1$, $p2$

1. v is a variable,
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)P(x)$	Premise
2.	$(\forall x)(P(x) \to Q(x))$	Premise
	$\vdots$	
n.	$(\exists x)Q(x)$	Goal

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1.	$P(a) \to Q(a)$	Premise
2.	$(\exists x)P(x)$	Premise
3.	$P(a)$	Assumption
4.	$Q(a)$	$\to$E : 1, 3
5.	$Q(a)$	$\exists$E : 2, 4, *BAD*!

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1.	$(\exists x)(\forall y)R(x, y)$	Premise
2.	$(\forall y)R(y, y)$	Assumption
3.	$R(a, a)$	$\forall$E : 1, 3
4.	$R(a, a)$	$\exists$E : 1, 3, *BAD*!

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2.	$(\forall y)R(y, y)$	Assumption
3.	$R(a, a)$	$\forall$E: 1, 3
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2.	$(\exists x)Q(x)$	Premise
3.	$P(v)$	Assumption
4.	$Q(v)$	Assumption
5.	$P(v)$ & $Q(v)$	&I : 3, 4
6.	$(\exists x)(P(x)$ & $Q(x))$	$\exists$I : 5
7.	$(\exists x)(P(x)$ & $Q(x))$	$\exists$E : 2, 6 *BAD*!
8.	$(\exists x)(P(x)$ & $Q(x))$	$\exists$E : 1, 7

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2.	$(\exists x)Q(x)$	Premise
3.	$P(v)$	Assumption
4.	$Q(v)$	Assumption
5.	$P(v)\ \&\ Q(v)$	$\&I: 3, 4$
6.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists I: 5$
7.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists E: 2, 6$ *BAD*!
8.	$(\exists x)(P(x)\ \&\ Q(x))$	$\exists E: 1, 7$

Universal Introduction ($\forall$I)

$p1.$	$\varphi[v/u]$	
	$\vdots$	
c.	$(\forall u)\varphi$	$\forall\mathrm{I} : p1$

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2. v does not occur in $(\forall u)\varphi$
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1.	$(\forall x)A(x)$	Premise
2.	$(\forall x)B(x)$	Premise
	$\vdots$	
n.	$(\forall x)(A(x) \mathbin{\&} B(x))$	Goal

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1.	$(\forall x)P(x,x)$	Premise
2.	$P(w,w)$	$\forall$E:1
3.	$(\forall x)P(x,w)$	&I:2BAD!

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2.	$P(w,w)$	$\forall$E : 1
3.	$(\forall x)P(x,w)$	&I : 2*BAD*!
4.	$(\forall y)(\forall x)P(x,y)$	&I : 3*BAD*!

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2.	$P(w,w)$	$\forall$E : 1
3.	$(\forall x)P(x,x)$	&I : 2

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2.	$(\forall y)P(w,y)$	$\forall$E : 1
3.	$P(w,w)$	$\forall$E : 2
4.	$(\forall x)P(x,x)$	$\forall$I : 3