

Introduction to Logic

PHIL 170

Eric Pacuit

University of Maryland, College Park

`pacuit.org`

`epacuit@umd.edu`

November 23, 2015

Announcements

- ▶ read Chapter 11 (and Chapter 12).
- ▶ Do the practice problems!
- ▶ Schedule
 1. Mon., 11/23: Ch. 11 (Deductions for Predicate Logic)
 2. Wed., 11/25: Ch. 11 (Deductions for Predicate Logic)
 3. Mon., 11/30: Ch. 11 (Deductions for Predicate Logic)
 4. Wed., 12/2: Ch. 11, course summary
 5. Mon., 12/7: Concluding remarks (general perspectives)
 6. Wed., 12/9: Concluding remarks (general perspectives)
- ▶ **Midterm #3:** Quiz (12/1, 11.59pm), Lab - Truth-Trees and Lab - Derivations (12/2, 11.59pm)
- ▶ **Final Exam:** Wed., Dec 16, 8:00am - 10:00am, LEF 2205

Final Exam

- ▶ Basic concepts: Logical connectives, tautology, contradictory, contingent formula, valid/invalid arguments, truth-value function, interpretations
- ▶ Translations into predicate logic
- ▶ Determine if a pair of formulas are logically equivalent.
- ▶ Determine if an argument is valid or invalid. Find a counterexample if the argument is not valid.
 - Using truth-tables (you need to know the truth-table rules)
 - Using truth-trees (the decompositions rules will be provided)
- ▶ Find deductions in propositional logic/predicate logic (deduction rules will be provided).

- ▶ **Tautology:** A formula of predicate logic is a tautology just in case it is true on every interpretation.
- ▶ **Contradictory Formula:** A formula of predicate logic is a contradictory just in case it is false on every interpretation.
- ▶ **Contingent Formula:** A formula of predicate logic is contingent just in case it is true on some interpretations, and false on others.

- ▶ **Tautology:** A formula of predicate logic is a tautology just in case it is true on every [interpretation](#).
- ▶ **Contradictory Formula:** A formula of predicate logic is a contradictory just in case it is false on every [interpretation](#).
- ▶ **Contingent Formula:** A formula of predicate logic is contingent just in case it is true on some [interpretations](#), and false on others.

An argument of predicate logic is **quantificationally valid** just in case there is no interpretation that makes all the premises of the argument true and the conclusion false.

An argument of predicate logic is **quantificationally valid** just in case there is no **interpretation** that makes all the premises of the argument true and the conclusion false.

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

► $P(x)[u/x] =$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

► $P(x)[u/x] = P(u)$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

- ▶ $P(x)[u/x] = P(u)$
- ▶ $(\forall x)P(x)[u/x] =$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

- ▶ $P(x)[u/x] = P(u)$
- ▶ $(\forall x)P(x)[u/x] = (\forall x)P(x)$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

- ▶ $P(x)[u/x] = P(u)$
- ▶ $(\forall x)P(x)[u/x] = (\forall x)P(x)$
- ▶ $(\forall x)R(x, y)[u/y] =$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

- ▶ $P(x)[u/x] = P(u)$
- ▶ $(\forall x)P(x)[u/x] = (\forall x)P(x)$
- ▶ $(\forall x)R(x, y)[u/y] = (\forall x)R(x, u)$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

- ▶ $P(x)[u/x] = P(u)$
- ▶ $(\forall x)P(x)[u/x] = (\forall x)P(x)$
- ▶ $(\forall x)R(x, y)[u/y] = (\forall x)R(x, u)$
- ▶ $((\forall x)P(x) \ \& \ R(x, y))[u/x] =$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

- ▶ $P(x)[u/x] = P(u)$
- ▶ $(\forall x)P(x)[u/x] = (\forall x)P(x)$
- ▶ $(\forall x)R(x, y)[u/y] = (\forall x)R(x, u)$
- ▶ $((\forall x)P(x) \ \& \ R(x, y))[u/x] = ((\forall x)P(x) \ \& \ R(u, y))$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

- ▶ $P(x)[u/x] = P(u)$
- ▶ $(\forall x)P(x)[u/x] = (\forall x)P(x)$
- ▶ $(\forall x)R(x, y)[u/y] = (\forall x)R(x, u)$
- ▶ $((\forall x)P(x) \ \& \ R(x, y))[u/x] = ((\forall x)P(x) \ \& \ R(u, y))$
- ▶ $(\forall x)R(x, y)[x/y] =$

Free, Bound Variables and Substitutions

An occurrence of a variable u in a formula is **bound** just in case that occurrence is in the scope of a quantifier that has u as its variable of quantification. An occurrence of a variable is **free** just in case it is not bound.

Substitution: $\varphi[u/x]$ is the formula φ in which every free occurrence of x is replaced with u .

Examples:

- ▶ $P(x)[u/x] = P(u)$
- ▶ $(\forall x)P(x)[u/x] = (\forall x)P(x)$
- ▶ $(\forall x)R(x, y)[u/y] = (\forall x)R(x, u)$
- ▶ $((\forall x)P(x) \ \& \ R(x, y))[u/x] = ((\forall x)P(x) \ \& \ R(u, y))$
- ▶ $(\forall x)R(x, y)[x/y] = (\forall x)R(x, x)$

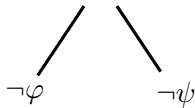
Truth-trees for predicate logic

$(\varphi \& \psi)$

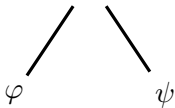
φ

ψ

$\neg(\varphi \& \psi)$



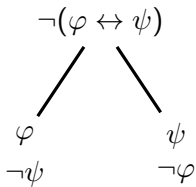
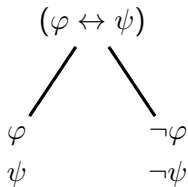
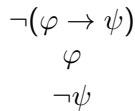
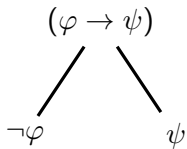
$(\varphi \vee \psi)$



$\neg(\varphi \vee \psi)$

$\neg\varphi$

$\neg\psi$



Decomposition Rules for Quantifiers

$$(\exists u)\varphi$$

$$\varphi[v/u]$$

$$(\forall u)\varphi$$

$$\varphi[t/u]$$

Provided v does not
appear on the branch

$$\neg(\exists u)\varphi$$

$$(\forall u)\neg\varphi$$

$$\neg(\forall u)\varphi$$

$$(\exists u)\neg\varphi$$

When is a branch completed?

For every decomposition rule, except the universal decomposition rule, when it is applied, check off the formula. *Universally quantified formulas are never checked off.*

admissible term: any constant or variable that has a **free occurrence** in a formula on the branch.

A truth-tree is **completed** once any formula on an open branch is either an atomic formula, the negation of an atomic formula, checked off, or a universally quantified formula that has been instantiated with every admissible term.

Which arguments are quantificationally valid?

$$\begin{array}{l} (\forall x)(H(x) \rightarrow M(x)) \\ 1. \quad H(s) \\ \hline \therefore M(s) \end{array}$$

$$\begin{array}{l} (\forall x)(H(x) \rightarrow M(x)) \\ 2. \quad \neg H(a) \\ \hline \therefore \neg M(a) \end{array}$$

$$\begin{array}{l} (\exists x)(S(x) \& C(x)) \\ 3. \quad (\exists x)(S(x) \& D(x)) \\ \hline \therefore (\exists x)(S(x) \& (D(x) \& C(x))) \end{array}$$

$$\begin{array}{l} (\forall x)(P(x) \rightarrow Q(x)) \\ 4. \quad (\forall x)(Q(x) \rightarrow R(x)) \\ \hline \therefore (\forall x)(P(x) \rightarrow R(x)) \end{array}$$

Deduction for Predicate Logic

Conjunction Introduction (&I)

$p1.$ φ

$p2.$ ψ

\vdots

$c.$ $(\varphi \ \& \ \psi)$ $\&I: p1, p2$

Conjunction Elimination (&EL, &ER)

$p1.$	$(\varphi \ \& \ \psi)$	
	\vdots	
c.	φ	$\&EL : p1$

$p1.$	$(\varphi \ \& \ \psi)$	
	\vdots	
c.	ψ	$\&ER : p1$

Conditional Introduction (\rightarrow I)

a1.	φ	Assumption
	\vdots	
p1.	ψ	Goal
c.	$(\varphi \rightarrow \psi)$	\rightarrow I: p1

p1.	ψ
	\vdots
c.	$(\varphi \rightarrow \psi) \rightarrow$ I: p1

Conditional Elimination (\rightarrow E)

$p1.$ φ

$p2.$ $(\varphi \rightarrow \psi)$

\vdots

$c.$ ψ $\rightarrow E: p1, p2$

Disjunction Introduction (\vee IL, \vee IR)

$p1.$	φ
\vdots	
c.	$(\varphi \vee \psi) \quad \vee\text{IR}: p1$

$p1.$	φ
\vdots	
c.	$(\psi \vee \varphi) \quad \vee\text{IL}: p1$

Disjunction Elimination ($\vee E$)

$p1.$	$(\varphi \vee \psi)$	Premise
$a1.$	φ	Assumption
	\vdots	
$p2.$	ρ	Goal
$a2.$	ψ	Assumption
	\vdots	
$p3.$	ρ	Goal
$c.$	ρ	$\vee E: p1, p2, p3$

Negation Introduction/Elimination ($\neg I$, $\neg E$)

a1.	φ	Assumption
	\vdots	
p1.	\perp	Goal
c.	$\neg\varphi$	$\neg I : p1$

a1.	$\neg\varphi$	Assumption
	\vdots	
p1.	\perp	Goal
c.	φ	$\neg E : p1$

Falsum Introduction/Elimination ($\perp I, \perp E$)

$p1.$	φ
$p2.$	$\neg\varphi$
	\vdots
c.	$\perp \quad \perp I: p1, p2$

$p1.$	\perp
	\vdots
c.	$\varphi \quad \perp E: p1$

Biconditional Introduction (\leftrightarrow I)

a1.	φ	Assumption
	\vdots	
p1.	ψ	Goal
a2.	ψ	Assumption
	\vdots	
p2.	φ	Goal
c.	$(\varphi \leftrightarrow \psi)$	\leftrightarrow I: p1, p2

p1.	φ
p2.	ψ
	\vdots
c.	$(\varphi \leftrightarrow \psi) \quad \leftrightarrow$ I: p1, p2

Biconditional Elimination (\leftrightarrow E)

$p1.$	$(\varphi \leftrightarrow \psi)$	
$p2.$	φ	
	\vdots	
$c.$	ψ	$\leftrightarrow E : p1, p2$

$p1.$	$(\varphi \leftrightarrow \psi)$	
$p2.$	ψ	
	\vdots	
$c.$	φ	$\leftrightarrow E : p1, p2$

Universal Elimination ($\forall E$)

$p1. (\forall x)\varphi$

\vdots

c. $\varphi[\tau/x] \quad \forall E: p1$

Universal Elimination ($\forall E$), Examples

1.	$(\forall x)(\forall y)R(x, y)$	Premise
2.	$(\forall u)(\forall v)Q(v, u)$	Premise
	\vdots	
$n.$	$R(a, b) \ \& \ Q(t, s)$	Goal

Universal Elimination ($\forall E$), Examples

1.	$(\forall x)(\forall y)R(x, y)$	Premise
2.	$(\forall u)(\forall v)Q(v, u)$	Premise
	\vdots	
$n.$	$R(a, b) \ \& \ Q(t, s)$	Goal

1.	$(\forall x)(P(x) \rightarrow Q(x, x))$	Premise
2.	$\neg Q(b, b)$	Premise
	\vdots	
$n.$	$\neg(\forall x)P(x)$	Goal

Existential Introduction ($\exists I$)

$p1. \quad \varphi[\tau/x]$

\vdots

c. $(\exists x)\varphi \quad \exists I: p1$

1.	$(\forall x)P(x)$	Premise
----	-------------------	---------

\vdots

$n.$	$(\exists x)P(x)$	Goal
------	-------------------	------

1.	$(\forall z)R(z, z)$	Premise
----	----------------------	---------

2.	$(\forall y)\neg O(y, y)$	Premise
----	---------------------------	---------

\vdots

$n.$	$(\exists x)(\exists y)(R(x, y) \ \& \ \neg O(y, x))$	Goal
------	---	------

$p1.$	$(\exists u)\varphi$	
$a1.$	$\varphi[v/u]$	Assumption
	\vdots	
$p2.$	ψ	Goal
$c.$	ψ	$\exists E: p1, p2$

1. v is a variable,
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)P(x)$	Premise
2.	$(\forall x)(P(x) \rightarrow Q(x))$	Premise
	\vdots	
$n.$	$(\exists x)Q(x)$	Goal

1. v is a variable
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$P(a) \rightarrow Q(a)$	Premise
2.	$(\exists x)P(x)$	Premise
3.	<div>$P(a)$</div>	Assumption
4.	<div>$Q(a)$</div>	$\rightarrow E: 1, 3$
5.	$Q(a)$	$\exists E: 2, 4, BAD!$

1. v is a variable
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$P(a) \rightarrow Q(a)$	Premise
2.	$(\exists x)P(x)$	Premise
3.	<div style="border: 1px solid black; padding: 2px;">$P(a)$</div>	Assumption
4.	<div style="border: 1px solid black; padding: 2px;">$Q(a)$</div>	$\rightarrow E: 1, 3$
5.	$Q(a)$	$\exists E: 2, 4, \text{BAD!}$

1. v is a variable
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)(\forall y)R(x, y)$	Premise
2.	$(\forall y)R(y, y)$	Assumption
3.	$R(a, a)$	$\forall E: 1, 3$
4.	$R(a, a)$	$\exists E: 1, 3, \text{BAD!}$

1. v is a variable
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)(\forall y)R(x, y)$	Premise
2.	$(\forall y)R(y, y)$	Assumption
3.	$R(a, a)$	$\forall E: 1, 3$
4.	$R(a, a)$	$\exists E: 1, 3, \text{BAD!}$

1. v is a variable
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)P(x)$	Premise
2.	$(\exists x)Q(x)$	Premise
3.	$P(v)$	Assumption
4.	$Q(v)$	Assumption
5.	$P(v) \& Q(v)$	$\&I: 3, 4$
6.	$(\exists x)(P(x) \& Q(x))$	$\exists I: 5$
7.	$(\exists x)(P(x) \& Q(x))$	$\exists E: 2, 6$ <i>BAD!</i>
8.	$(\exists x)(P(x) \& Q(x))$	$\exists E: 1, 7$

1. v is a variable
2. v does not occur in φ
3. v does not occur in ψ
4. v does not occur free in any assumption on which line $p2$ depends, except, of course, in $a1$.

1.	$(\exists x)P(x)$	Premise
2.	$(\exists x)Q(x)$	Premise
3.	$P(v)$	Assumption
4.	$Q(v)$	Assumption
5.	$P(v) \& Q(v)$	$\&I: 3, 4$
6.	$(\exists x)(P(x) \& Q(x))$	$\exists I: 5$
7.	$(\exists x)(P(x) \& Q(x))$	$\exists E: 2, 6$ BAD!
8.	$(\exists x)(P(x) \& Q(x))$	$\exists E: 1, 7$

Universal Introduction ($\forall I$)

$p1.$	$\varphi[v/u]$
\vdots	
$c.$	$(\forall u)\varphi \quad \forall l: p1$

1. v is a variable,
2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)A(x)$	Premise
2.	$(\forall x)B(x)$	Premise
	\vdots	
$n.$	$(\forall x)(A(x) \ \& \ B(x))$	Goal

1. v is a variable,
2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)P(x, x)$	Premise
2.	$P(w, w)$	$\forall E: 1$
3.	$(\forall x)P(x, w)$	$\&I: 2BAD!$

1. v is a variable,
2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)P(x, x)$	Premise
2.	$P(w, w)$	$\forall E: 1$
3.	$(\forall x)P(x, w)$	$\&I: 2BAD!$
4.	$(\forall y)(\forall x)P(x, y)$	$\&I: 3BAD!$

1. v is a variable,
2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)P(x, x)$	Premise
2.	$P(w, w)$	$\forall E: 1$
3.	$(\forall x)P(x, x)$	$\&I: 2$

1. v is a variable,
2. v does not occur in $(\forall u)\varphi$
3. v does not occur free in any assumption on which line $p1$ depends.

1.	$(\forall x)(\forall y)P(x, y)$	Premise
2.	$(\forall y)P(w, y)$	$\forall E: 1$
3.	$P(w, w)$	$\forall E: 2$
4.	$(\forall x)P(x, x)$	$\forall I: 3$