Logical Notation

Notes for PHIL 470

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1 First-Order Language

The language of predicate logic is constructed from a number of different pieces of syntax: variables, constants, function symbols and predicate symbols. Both function and predicate symbols are associated with an *arity*: the number of arguments that are required by the function or predicate. We start by defining **terms**. Let $\mathcal{V}$ be a finite (or countable) set of **variables** and $\mathcal{C}$ a set of **constants**.

Definition 1.1 (Terms) Let $\mathcal{V}$ be a set of variable, $\mathcal{C}$ a set of constant symbols and $\mathcal{F}$ a set of function symbols. Each function symbol is associated with an **arity** (a positive integer specifying the number of arguments). Write $f^{(n)}$ if the arity of f is n. A term τ is constructed as follows:

- Any variable $x \in \mathcal{V}$ is a term.
- Any constant $c \in \mathcal{C}$ is a term.
- If $f^{(n)} \in \mathcal{F}$ is a function symbol (i.e., f accepts n arguments) and $\tau_1, \ldots, \tau_n$ are terms, then $f(\tau_1, \ldots, \tau_n)$ is a term.
- Nothing else is a term.

Let $\mathcal{T}$ be the set of terms. $\triangleleft$

The language of arithmetic. The language of arithmetic is constructed from a single constant $\mathcal{C} = \{\mathbf{0}\}$, the function symbols $\mathcal{F} = \{\mathbf{S}, +, *, \mathbf{E}\}$, where $\mathbf{S}$ is a unary function symbol and $+, *, \mathbf{E}$ are binary function symbols. Examples of terms in this language are $\mathbf{S}(\mathbf{S}(\mathbf{0}))$, $\mathbf{S}(x)$, $+(x, \mathbf{S}(\mathbf{S}(\mathbf{S}(\mathbf{0}))))$, $\mathbf{S}(+(*(x, z), \mathbf{S}(\mathbf{0})))$.

To increase readability, we typically use infix notation rather than prefix notation. So, we write $x + y$ instead of $+(x, y)$.

Terms are used to construct atomic formulas:

Definition 1.2 (Atomic Formulas) Let $\mathcal{P}$ be a set of predicate symbols. Each predicate symbol is associated with an arity (the number of objects that are related by P). We write $P^{(n)}$ if the arity of P is n. Suppose that P is an atomic predicate symbol with arity n. If $\tau_1, \ldots, \tau_n$ are terms, then $P(\tau_1, \ldots, \tau_n)$ is an atomic formula. To simplify the notation, we may write $P\tau_1\tau_2 \cdots \tau_n$. A special predicate symbol '=' is included with the intended interpretation *equality.* $\triangleleft$

The language of arithmetic. The language of arithmetic includes two predicate symbols: equality = and less-than <. Both are binary relation symbols. Again, we use infix notation to increase readability. Examples of formulas include $\mathbf{S}(x) = \mathbf{S}(\mathbf{S}(\mathbf{S}(\mathbf{0})))$, $(x + \mathbf{S}(y * \mathbf{S}(\mathbf{S}(\mathbf{S}(\mathbf{0})))) < \mathbf{S}(z)$, $\mathbf{S}(\mathbf{S}(\mathbf{0})) = \mathbf{S}(\mathbf{0})$.

Definition 1.3 (Formulas) Formulas are constructed as follows:

- Atomic formulas $P(\tau_1, \ldots, \tau_n)$ are formulas;
- If φ is a formula, then so is $\neg\varphi$;
- If φ and ψ are a formulas, then so is $\varphi \wedge \psi$;
- If φ is a formula, then so is $(\forall x)\varphi$, where x is a variable;
- Nothing else is a formula.

The other boolean connectives $(\vee, \rightarrow, \leftrightarrow)$ are defined as usual. In addition, $(\exists x)\varphi$ is defined as $\neg(\forall x)\neg\varphi$. $\triangleleft$

The language of arithmetic. Examples of formulas in the language of arithmetic include: $\neg(\mathbf{S}(\mathbf{0}) = \mathbf{S}(\mathbf{S}(\mathbf{0}))$ (this is usually written as $\mathbf{S}(\mathbf{0}) \neq \mathbf{S}(\mathbf{S}(\mathbf{0}))$), $\forall x \mathbf{S}(\mathbf{S}(\mathbf{S}(\mathbf{S}(\mathbf{S}(x))))) \neq x$,
$\forall x(x \neq \mathbf{0} \rightarrow \exists y(\mathbf{S}(y) = x)$

Definition 1.4 (Free Variable) Suppose that x is a variable. Then, x **occurs free in** φ is defined as follows:

1. If φ is an atomic formula, then x occurs free in φ provided x occurs in φ (i.e., is a symbol in φ).
2. x occurs free in $\neg\psi$ iff x occurs free in ψ
3. x occurs free in $\psi_1 \wedge \psi_2$ iff x occurs free in ψ_1 or x occurs free in ψ_2
4. x occurs free in $(\forall y)\psi$ iff x occurs free in ψ and $x \neq y$
5. x occurs free in $(\exists y)\psi$ iff x occurs free in ψ and $x \neq y$ $\triangleleft$

The set of free variables in φ, denoted $\mathsf{Fr}(\varphi)$, is defined by recursion as follows:

1. If φ is an atomic formula, then $\mathsf{Fr}(\varphi)$ is the set of all variables (if any) that occur in φ
2. If φ is $\neg\psi$, then $\mathsf{Fr}(\neg\varphi) = \mathsf{Fr}(\varphi)$
3. If φ is $\psi_1 \wedge \psi_2$, then $\mathsf{Fr}(\varphi) = \mathsf{Fr}(\psi_1) \cup \mathsf{Fr}(\psi_2)$

4. If φ is $(\forall x)\psi$, then $\mathsf{Fr}(\psi) = \mathsf{Fr}(\psi)$ after removing x, if present.

A variable x that is not free is said to be **bound**. Formulas that do not contain any free variables are called sentences:

Definition 1.5 (Sentence) If φ is a formula and $\mathsf{Fr}(\varphi) = \emptyset$ (i.e., there are no free variables), then φ is a **sentence**. $\triangleleft$

1.1 Substitutions

If τ and τ' are terms, we write $\tau[x/\tau']$ for the terms where x is replaced by τ'. We can formally define this operation by recursion:

- $x[x/\tau'] = \tau'$
- $y[x/\tau'] = y$ for $x \neq y$
- $c[x/\tau'] = c$
- $F(\tau_1, \ldots, \tau_n)[x/\tau'] = F(\tau_1[x/\tau'], \ldots, \tau_n[x/\tau'])$

The same notation can be used for formulas $\varphi[x/\tau]$ which means replace all free occurrences of x with τ in a formula φ. This is defined as follows:

- $P(\tau_1, \ldots, \tau_n)[x/\tau] = P(\tau_1[x/\tau], \ldots, \tau_n[x/\tau])$
- $\neg\psi[x/\tau] = \neg(\varphi[x/\tau])$
- $(\psi_1 \wedge \psi_2)[x/\tau] = \psi_1[x/\tau] \wedge \psi_2[x/\tau]$
- $((\forall x)\varphi)[x/\tau] = (\forall x)\varphi$
- $((\forall y)\varphi)[x/\tau] = (\forall y)\varphi[x/\tau]$, where $y \neq x$

The following are key examples of this operation:

1. $(x = y)[y/x]$ is $x = x$ and $(x = y)[x/y]$ is $y = y$,
2. $(\forall x(x = y))[x/y]$ is $(\forall x)x = y$,
3. $(\forall x(x = y))[y/x]$ is $(\forall x)x = x$,
4. $(\forall x)\neg(\forall y)(x = y) \to (\neg\forall y(x = y))[x/y]$ is $(\forall x)\neg(\forall y)(x = y) \to \neg\forall y(y = y)$.

Definition 1.6 (Substitutability) A term τ is **substitutable for x in** φ is defined as follows:

- For an atomic formula φ, τ is always substitutable for x in φ (there are no quantifiers, so t can always be substituted for x)
- τ is substitutable for x in $\neg\psi$ iff τ is is substitutable for x in ψ
- τ is substitutable for x in $\psi_1 \wedge \psi_1$ iff τ is is substitutable for x in ψ_1 and τ is is substitutable for x in ψ_2
- τ is substitutable for x in $(\forall y)\psi$ iff either
    1. x does not occur free in $(\forall y)\psi$
    2. y does not occur in τ and τ is substitutable for x in ψ. $\triangleleft$

2 Models

2.1 Interpreting Terms

Suppose that W is a set. An **interpretation** I (for W) associates with each functions symbol F a function on W of the appropriate arity, denoted F^I, and to each constant c an element of W, denoted c^I. If W is a set and I an interpretation, then for a function symbol F of arity n,

$$F^I : \underbrace{W \times \cdots \times W}_{n \text{ times}} \to W$$

For each constant symbol, c, we have

$$c^I \in W$$

Our goal is to show how to associate with each term and element of a set W. We first need the notion of a substitution:

Definition 2.1 (Substitution) Suppose that W is a nonempty set. A **substitution** is a function $\mathbf{s} : \mathcal{V} \to W$. ◁

Definition 2.2 (Interpretation of Terms) Suppose that I is an interpretation for W and $\mathbf{s} : \mathcal{V} \to W$ is a substitution. We define the function $(I, \mathbf{s}) : \mathcal{T} \to W$ by recursion as follows:

- $(I, \mathbf{s})(x) = \mathbf{s}(x)$
- $(I, \mathbf{s})(c) = c^I$
- $(I, \mathbf{s})(F(\tau_1, \ldots, \tau_n)) = F^I((I, \mathbf{s})(\tau_1), \ldots, (I, \mathbf{s})(\tau_n))$ ◁

Suppose that $\mathbf{s} : \mathcal{V} \to W$ is a substitution. If $a \in W$, we define a new substitution $\mathbf{s}[x/a]$ as follows:

$$\mathbf{s}[x/a](y) = \begin{cases} a & \text{if } y = x \\ \mathbf{s}(x) & \text{otherwise} \end{cases}$$

Suppose that $\mathbf{s} : \mathcal{V} \to W$ and $\mathbf{s}' : \mathcal{V} \to W$ are two substitutions. For each variable $x \in \mathcal{V}$, we define a relation on the set of substitutions as follows:

$$\mathbf{s} \sim_x \mathbf{s}' \text{ iff } \mathbf{s}(y) = \mathbf{s}'(y) \text{ for all } y \neq x$$

Hence, $\mathbf{s} \sim_x \mathbf{s}'$ provided there is some $a \in W$ such that $\mathbf{s}' = \mathbf{s}[x/a]$.

2.2 First Order Models

Definition 2.3 (Model) A model is a pair $\mathfrak{A} = \langle W, I \rangle$ where W is a nonempty set (called the domain) and I is a function (called the interpretation) assigning to each function symbol F, a function denoted F^I, to each constant symbol, an element of W denoted c^I and to each predicate symbol P, a relation on W of the appropriate arity. If P has arity n, then we have

$$P^I \subseteq \underbrace{W \times \cdots \times W}_{n \text{ times}}$$

If $\mathcal{A}$ is a model, we write $|\mathcal{A}|$ for the domain of $\mathcal{A}$, and we write $F^{\mathcal{A}}$, $c^{\mathcal{A}}$ and $P^{\mathcal{A}}$ to denote F^I, c^I and P^I, respectively. ◁

We say $\mathbf{s}$ is a substitution for $\mathcal{A}$ provided $\mathbf{s} : \mathcal{V} \to |\mathcal{A}|$. Let $\mathcal{A} = \langle W, I \rangle$ be a model. For each term τ, we write $\tau^{\mathcal{A},\mathbf{s}}$ for $(I, \mathbf{s})(\tau)$.

Definition 2.4 (Truth) Suppose that $\mathcal{A}$ is a model and $\mathbf{s}$ is a substitution for $\mathcal{A}$. The formula φ is true in $\mathcal{A}$ (given $\mathbf{s}$), denoted $\mathcal{A}, \mathbf{s}\varphi$, is defined by recursion as follows:

- $\mathcal{A}, \mathbf{s} \models P(\tau_1, \ldots, \tau_n)$ iff $(\tau_1^{\mathcal{A},\mathbf{s}}, \ldots, \tau_n^{\mathcal{A},\mathbf{s}}) \in P^{\mathcal{A}}$
- $\mathcal{A}, \mathbf{s} \models \neg\psi$ iff $\mathcal{A}, \mathbf{s} \not\models \psi$
- $\mathcal{A}, \mathbf{s} \models \psi_1 \wedge \psi_2$ iff $\mathcal{A}, \mathbf{s} \models \psi_1$ and $\mathcal{A}, \mathbf{s} \models \psi_2$
- $\mathcal{A}, \mathbf{s} \models (\forall x)\psi$ iff for all substitutions $\mathbf{s}'$ for $\mathcal{A}$ if $\mathbf{s} \sim_x \mathbf{s}'$, then $\mathcal{A}, \mathbf{s}' \models \psi$ $\lhd$

Models of arithmetic Recall that $\mathbb{N}$ denotes the set of natural numbers (i.e., the integers greater than or equal to 0). We will be interested in the following languages and models

- $\mathcal{N}_S = (\mathbb{N}, \mathbf{0}, \mathbf{S})$ with the language $\mathcal{L}_S$ constructed from the constant $\mathbf{0}$ and function symbol $\mathbf{S}$ (and the equality symbol). So, $\mathcal{L}_S$ is a subset of the language of arithmetic $\mathcal{L}_A$.
- $\mathcal{N}_P = (\mathbb{N}, \mathbf{0}, \mathbf{S}, +)$ with the language $\mathcal{L}_P$ constructed from the constant $\mathbf{0}$ and function symbols $\mathbf{S}$ and $+$ (and the equality symbol). So, $\mathcal{L}_P$ is a subset of the language of arithmetic $\mathcal{L}_A$
- $\mathcal{N} = (\mathbb{N}, \mathbf{0}, \mathbf{S}, +, *, <)$ with the language of arithmetic $\mathcal{L}_A$.

3 Deductions in First Order Logic

An axiom system for first-order logic consists of the following four axioms (there are others, this is the one from Enderton's *Introduction to Mathematical Logic*):

1. All tautologies
2. $(\forall x)\varphi \to \varphi[x/t]$, where τ is substitutable for x in φ
3. $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$
4. $\varphi \to (\forall x)\varphi$, where x does not occur free in φ

Definition 3.1 (Generalization) Given a formula φ, a **generalization of** φ is a formula of the form $(\forall x_1) \cdots (\forall x_n)\varphi$. $\lhd$

Definition 3.2 (Tautology) A tautology (in FOL) is any formula obtained by replacing each atomic proposition with a first-order formula. $\lhd$

Definition 3.3 (Deduction) We write $\Gamma \vdash \varphi$ iff there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$, each φ_i is either a generalization of one of the above axioms, is an element of Γ, or follows from earlier formulas on the list by modus ponens. We write $\vdash \varphi$ instead of $\emptyset \vdash \varphi$. $\lhd$

Example . $\vdash \exists x(\alpha \wedge \beta) \rightarrow \exists x\alpha \wedge \exists x\beta$.

1.	$\forall x(\neg\alpha \rightarrow \neg(\alpha \wedge \beta))$	Instance of Axiom 1
2.	$\forall x(\neg\alpha \rightarrow \neg(\alpha \wedge \beta)) \rightarrow (\forall x\neg\alpha \rightarrow \forall x\neg(\alpha \wedge \beta))$	Instance of Axiom 3
3.	$\forall x\neg\alpha \rightarrow \forall x\neg(\alpha \wedge \beta)$	MP 1,2
4.	$(\forall x\neg\alpha \rightarrow \forall x\neg(\alpha \wedge \beta)) \rightarrow (\neg\forall x\neg(\alpha \wedge \beta) \rightarrow \neg\forall x\neg\alpha)$	Instance of Axiom 1
5.	$\neg\forall x\neg(\alpha \wedge \beta) \rightarrow \neg\forall x\neg\alpha$	MP 3,4
6.	$\exists x(\alpha \wedge \beta) \rightarrow \exists x\alpha$	Definition of '$\exists$'
7.	$\forall x(\neg\beta \rightarrow \neg(\alpha \wedge \beta))$	Instance of Axiom 1
8.	$\forall x(\neg\beta \rightarrow \neg(\alpha \wedge \beta)) \rightarrow (\forall x\neg\beta \rightarrow \forall x\neg(\alpha \wedge \beta))$	Instance of Axiom 3
9.	$\forall x\neg\beta \rightarrow \forall x\neg(\alpha \wedge \beta)$	MP 7,8
10.	$(\forall x\neg\beta \rightarrow \forall x\neg(\alpha \wedge \beta)) \rightarrow (\neg\forall x\neg(\alpha \wedge \beta) \rightarrow \neg\forall x\neg\beta)$	Instance of Axiom 1
11.	$\neg\forall x\neg(\alpha \wedge \beta) \rightarrow \neg\forall x\neg\beta$	MP 9,10
12.	$\exists x(\alpha \wedge \beta) \rightarrow \exists x\beta$	Definition of '$\exists$'
13.	$(\exists x(\alpha \wedge \beta) \rightarrow \exists x\alpha) \rightarrow ((\exists x(\alpha \wedge \beta) \rightarrow \exists x\beta)$ $\rightarrow (\exists x(\alpha \wedge \beta) \rightarrow (\exists x\alpha \wedge \exists x\beta)))$	Instance of Axiom 1
14.	$(\exists x(\alpha \wedge \beta) \rightarrow \exists x\beta) \rightarrow (\exists x(\alpha \wedge \beta) \rightarrow (\exists x\alpha \wedge \exists x\beta))$	MP 6,13
15.	$\exists x(\alpha \wedge \beta) \rightarrow (\exists x\alpha \wedge \exists x\beta)$	MP 12, 14

4 Basic Model Theory

- A set of formulas T is **inconsistent** provided $T \vdash \bot$ (where $\bot$ is a formula of the form $\mathbf{0} \neq \mathbf{S}(\mathbf{0})$. A set of formulas T is **consistent** if it is not inconsistent.

- Suppose that T is a set of sentences. Then $Cn(T) = \{\varphi \mid T \vdash \varphi\}$ is the set of (first-order) **consequences** of T.

- Suppose that $\mathcal{A}$ is a first-order model. Then, $Th(\mathcal{A}) = \{\varphi \mid \varphi \text{ is a sentence and } \mathcal{A} \models \varphi\}$ is the **theory of** $\mathcal{A}$. For example, $Th(\mathcal{N}_S)$ is the set of sentences of $\mathcal{L}_S$ true in $\mathcal{N}_S$; and $Th(\mathcal{N})$ is the set of sentences of $\mathcal{L}_A$ true in $\mathcal{N}$ (the **theory of true arithmetic**).

- A set of sentences T is **satisfiable** if there is a model $\mathcal{A}$ such that $\mathcal{A} \models T$ (where $\mathcal{A} \models T$ means $\mathcal{A} \models \varphi$ for each $\varphi \in T$).

- A **theory** is a set of sentences. (Sometimes

  A **theory** is (effectively) axiomatizable provided there is recursive set A of sentences (and possibly rules) such that $Cn(A) = T$. A theory T is **finitely axiomatizable** provided there is a finite set A of sentences (and possibly rules) such that $Cn(A) = T$.

  A theory T (in the language $\mathcal{L}$) is **negation-complete** provided for every sentence of φ in $\mathcal{L}$, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

  A theory T is **decidable** provided the set $Cn(T)$ is recursive.

Some useful observations and Theorems:

- If $\mathcal{L}$ is a first-order language constructed from a signature of size κ (where κ is a cardinal), then $|\mathcal{L}| = \max\{\aleph_0, \kappa\}$ ($\aleph_0$ is the first countable cardinal). Thus, there are countably many formulas of $\mathcal{L}_A$.

- The set $\mathcal{L}$ of well-formed formulas (wff) is recursive.

- If T is effectively axiomatizable, then $Cn(T)$ is semidecidable.

- If T is effectively axiomatizable and negation-complete, then $Cn(T)$ is decidable.

- *Model Construction Theorem.* Every consistent set of formulas has a model.

- *Compactness Theorem.* If every finite subset of T is satisfiable, then T is satisfiable.

- *Löwenheim-Skolem Theorem.* If T has a model, then T has a countable model. A model $\mathcal{A}$ is countable provided the domain of $\mathcal{A}$ is countable (i.e., $|\mathcal{A}|$ is countable). The upward Löwenheim-Skolem Theorem states that if T has a model, then it has a model of any infinite cardinality κ.

Two structures $\mathcal{A}$ and $\mathcal{B}$ are **elementarily equivalent**, denoted $\mathcal{A} \equiv \mathcal{B}$, provided for every sentence φ, $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$ (i.e., $Th(\mathcal{A}) = Th(\mathcal{B})$).

Definition 4.1 (Isomorphism) Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two models. A function $f : |\mathcal{A}| \to |\mathcal{B}|$ is an **isomorphism** provided

- f is a bijection

- For all constants $c \in \mathcal{C}$, $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$

- $f(F^{\mathcal{A}}(a_1, \ldots, a_n)) = F^{\mathcal{B}}(f(a_1), \ldots, f(a_n))$

- For all $(a_1, \ldots, a_n) \in P^{\mathcal{A}}$ iff $(f(a_1), \ldots, f(a_n) \in P^{\mathcal{B}}$

We write $\mathcal{A} \cong \mathcal{B}$ when there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. $\triangleleft$

Isomorphism Theorem. For any two first-order models if $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$.

There are examples of structures that are elementarily equivalent but not isomorphic (e.g., $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$ cannot be distinguished by a first-order formula, but are not isomorphic since there is no bijection function from $\mathbb{R}$ to $\mathbb{Q}$.)

Suppose that $\mathcal{A}$ is a first-order structure. A set $X \subseteq |\mathcal{A}|$ is **definable** (in the language $\mathcal{L}$) provided there is a formula $\varphi(x)$ with one free variable such that

$$X = \{a \mid \mathcal{A} \models \varphi(a)\}$$

This definition can be readily adapted to k-ary relations $X \subseteq |\mathcal{A}|^k$.

Example. $\mathbb{N}$ is not definable in the structure $(\mathbb{R}, <)$. Suppose it is defined by $\varphi(x)$ in the first-order language with equality and $<$. Consider $h : \mathbb{R} \to \mathbb{R}$ defined as $h(r) = r^3$. Then, h is a isomorphism between $(\mathbb{R}, <)$ and itself (it is an *automorphism*). Thus, by the Isomorphism Theorem, $(\mathbb{R}, <) \models \varphi(r)$ iff $(\mathbb{R}, <) \models \varphi(h(r))$. But, then $\sqrt[3]{2} \notin \mathbb{N}$ implies $(\mathbb{R}, <) \not\models \varphi(\sqrt[3]{2})$ iff $(\mathbb{R}, <) \not\models \varphi(h(\sqrt[3]{2}))$ iff $(\mathbb{R}, <) \not\models \varphi(2)$, which is a contradiction since $2 \in \mathbb{N}$.

A theory T is κ**-categorical** (where κ is an infinite cardinal) provided every model of T of size κ is isomorphic.

Łoś-Vaught Test. Let T be a theory in a countable language. Assume that T has no finite models. If T is κ-categorical for some cardinal κ, then T is negation-complete.

Fact. If $T \subseteq T'$, T' is satisfiable and T is negation-complete, then $T = T'$.

A theory T satisfies **quantifier elimination** provided for all sentences φ, there is a quantifier-free formula ψ such that

$$T \vdash \varphi \leftrightarrow \psi$$

Theorem. Suppose that T is a theory such that for each sentence of the form $\exists x(\alpha_1 \wedge \cdots \wedge \alpha_k)$, where each α_i is a literal, there is a quantifier-free sentences ψ such that

$$T \vdash \exists x(\alpha_1 \wedge \cdots \wedge \alpha_k) \leftrightarrow \psi$$

Then, T satisfies quantifier elimination.

Example. The theory of natural numbers with successor, $Th(\mathcal{N}_S)$ where $\mathcal{N}_S = (\mathbb{N}, \mathbf{0}, \mathbf{S})$ satisfies quantifier elimination.