Chapter 3
Review of Paradoxes Afflicting Procedures for Electing a Single Candidate

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3.1 Introduction

Three factors motivated me to write this chapter:

- The recent passage (25 February 2010) by the British House of Commons of the *Constitutional Reform and Governance Bill*, clause #29 of which states that a referendum will be held by 31 October 2011 on changing the current *single member plurality* (aka *first-past-the-post*, briefly FPTP) electoral procedure for electing the British House of Commons to the (highly paradoxical) *alternative vote* (AV) procedure (aka *Instant Runoff*).[1] Similar calls for adopting the alternative vote procedure are voiced also in the US.
- My assessment that both the UK and the US will continue to elect their legislatures from single-member constituencies, but that there exist, from the point of view of social-choice theory, considerably more desirable voting procedures for electing a single candidate than the FPTP and AV procedures.
- A recent report by Hix et al. (2010) – commissioned by the British Academy and entitled *Choosing an Electoral System* – that makes no mention of standard social-choice criteria for assessing electoral procedures designed to elect one out of two or more candidates.

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[1] Following the general elections held in the UK on 6 May 2010, a coalition government has been formed between the Conservative and Liberal-Democratic parties in which the two parties committed to hold a referendum on the possible change of the election procedure to the House of Commons from FPTP to AV. In the referendum held on 5 May 2011 it was decided to keep the FPTP procedure.

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I therefore thought it would be well to supplement that report by reminding social choice theorists, political scientists, as well as commentators, policymakers and interested laymen – especially in the UK and the US – of the main social-choice properties by which voting procedures for the election of one out of two or more candidates ought to be assessed, and to list and exemplify the paradoxes afflicting these voting procedures.

Thus this paper should be regarded as an updated review by which to assess from a social-choice perspective the main properties of various known voting procedures for the election of a single candidate.

Of the 18 (deterministic) voting procedures analyzed in this paper, the Condorcet-consistent procedures proposed by Copeland (1951) and by Kemeny (1969) seem to me to be the most desirable from a social-choice perspective for electing one out of several candidates.

The paper is organized as follows: In Sect. 3.2 I survey 15 paradoxes, several of which may afflict any of the 18 voting procedures that are described in Sect. 3.3. Section 3.4 summarizes and presents additional *technical-administrative* criteria which should be used in assessing the relative desirability of a voting procedure. In the detailed appendix in Sect. 3.5 I exemplify most of the paradoxes to which each of the surveyed election procedures is susceptible.

3.2 Voting Paradoxes

I define a "*voting paradox*" as an undesirable outcome that a voting procedure may produce and which may be regarded at first glance, at least by some people, as surprising or as counter-intuitive.

I distinguish between two types of voting paradoxes associated with a given voting procedure:

1. "*Simple*" or "*Straightforward*" paradoxes: These are paradoxes where the relevant data leads to a "surprising" and arguably undesirable outcome. (The relevant data include, *inter alia*, the number of voters, the number of candidates, the number of candidates that must be elected, the preference ordering of every voter among the competing candidates, the amount of information voters have regarding all other voters' preference orderings, the order in which voters cast their votes if it is not simultaneous, the order in which candidates are voted upon if candidates are not voted upon simultaneously, whether voting is open or secret, the manner in which ties are to be broken).
2. "*Conditional*" *paradoxes*: These are paradoxes where changing one relevant datum while holding constant all other relevant data leads to a "surprising" and arguably undesirable outcome.

An array of paradoxes of one or both types are described and analyzed by McGarvey (1953), Riker (1958), Smith (1973), Fishburn (1974, 1977, 1981, 1982), Young (1974), Niemi and Riker (1976), Doron and Kronick (1977), Doron (1979), Richelson (1979), Gehrlein (1983), Fishburn and Brams (1983), Saari (1984, 1987, 1989, 1994, 2000, 2008), Niou (1987), Moulin (1988a), Merlin and Saari (1997), Brams, Kilgour and Zwicker (1998), Scarsini (1998), Nurmi (1998a, 1998b, 1999, 2004, 2007), Lepelley and Merlin (2001), Merlin et al. (2002), Merlin and Valognes (2004), Tideman (1987, 2006), Gehrlein and Lepelley (2011), among others.

3.2.1 Simple Paradoxes

The six best-known "simple" paradoxes that may afflict voting procedures designed to elect one out of two or more candidates are the following:

3.2.1.1 The Condorcet (or Voting, or Cyclical Majorities) Paradox (Condorcet 1785; Black 1958)

Given that the preference ordering of every voter among the competing candidates is transitive, the (amalgamated) preference ordering of the majority of voters among the competing candidates may nevertheless be intransitive. A necessary condition for this to occur is that the various majorities are composed of different persons and there exist at least three candidates. Although we do not demonstrate this paradox in the Appendix, it may occur under all ranked voting procedures, as well as under the successive elimination procedure.

3.2.1.2 The Condorcet Winner Paradox (Condorcet 1785; Black 1958)

A candidate x is not elected despite the fact that it constitutes a "Condorcet Winner", i.e., despite the fact that x is preferred by a majority of the voters over each of the other competing alternatives.[2]

3.2.1.3 The Absolute Majority Paradox

This is a special case of the Condorcet winner paradox. A candidate x may not be elected despite the fact that it is the only candidate ranked first by an absolute majority of the voters.

[2] Fishburn (1974, p. 544) constructs an example with 101 voters and nine candidates two of whom are candidates a and w, such that w beats each of the other eight candidates by a (slim) majority of 51 to 50 (and hence is a Condorcet winner), whereas a beats each of the other seven candidates by a considerably larger majority. Fishburn states that "examples like this suggest that some cases which have a simple-majority [Condorcet] winner do *not* represent the most satisfactory social choice." We disagree with this statement and hold that a Condorcet winner, if one exists, ought *always* to be elected.

3.2.1.4 The Condorcet Loser or Borda Paradox (Borda 1784; Black 1958)

A candidate x is elected despite the fact that it constitutes a "Condorcet Loser" i.e., despite the fact that a majority of voters prefer each of the remaining candidates to x. This paradox is a special case of the violation of Smith's (1973) Condorcet principle. According to this principle, if it is possible to partition the set of candidates into two disjoint subsets, A and B, such that each candidate in A is preferred by a majority of the voters over each candidate in B, then no candidate in B ought to be elected unless all candidates in A are elected.

3.2.1.5 The Absolute Loser Paradox

This is a special case of the Condorcet loser paradox. A candidate x may be elected despite the fact that it is ranked last by a majority of voters.

3.2.1.6 The Pareto (or Dominated Candidate) Paradox (Fishburn 1974)

A candidate x may be elected while candidate y may not be elected despite the fact that *all* voters prefer candidate y to x.

3.2.2 Conditional Paradoxes

The nine best-known "conditional" paradoxes that may afflict voting procedures for electing a single candidate are the following:

3.2.2.1 Additional Support (or Lack of Monotonicity or Negative Responsiveness) Paradox (Smith 1973; Fishburn 1974a, Fishburn and Brams 1983)

If candidate x is elected under a given distribution of voters' preferences among the competing candidates, it is possible that, *ceteris paribus*, x may not be elected if some voter(s) *increase(s) his (their) support for* x by moving x to a higher position in his (their) preference ordering. Alternatively, if candidate x is not elected under a given distribution of voters' preferences among the competing candidates, it is possible that, *ceteris paribus*, x will be elected if some voter(s) *decrease(s)* his (their) support for x by moving x to a lower position in his (their) preference ordering.[3]

[3] Another version of the non-monotonicity paradox (which is not demonstrated in the Appendix) is a situation where x is elected in a given electorate but may not be elected if, *ceteris paribus*, additional voters join the electorate who rank x at the top of their preference ordering, or,

3.2.2.2 Reinforcement (or Inconsistency or Multiple Districts) Paradox (Young 1974)

If x is elected in each of several disjoint electorates, it is possible that, *ceteris paribus*, x will not be elected if all electorates are combined into a single electorate.

3.2.2.3 Truncation Paradox (Brams 1982; Fishburn and Brams 1983)

A voter may obtain a more preferable outcome if, *ceteris paribus*, he lists in his ballot only part of his (sincere) preference ordering among some of the competing candidates than listing his entire preference ordering among all the competing candidates.

3.2.2.4 No-Show paradox (Fishburn and Brams 1983; Ray 1986; Moulin 1988b; Holzman 1988/89; Pérez 1995)

This is an extreme version of the truncation paradox. A voter may obtain a more preferable outcome if he decides not to participate in an election than, *ceteris paribus*, if he decides to participate in the election and vote sincerely for his top preference(s).

3.2.2.5 Twin Paradox (Moulin 1988b)

This is a special version of the no-show paradox. Two voters having the same preference ordering may obtain a preferable outcome if, *ceteris paribus*, one of them decides not to participate in the election while the other votes sincerely.

3.2.2.6 Violation of the Subset Choice Condition (SCC) (Fishburn 1974b,c, 1977)

SCC requires that when there are at least three candidates and candidate x is the unique winner, then x must not become a loser whenever any of the original losers is removed and all other things remain the same. All the voting procedures discussed in this paper except the range voting (RV) and majority judgment (MJ) procedures violate SCC.[4] In the context of individual choice theory SCC is known as Chernoff's

alternatively, a situation where x is not elected in a given electorate but may be elected if, *ceteris paribus*, additional voters join the electorate who rank x at the bottom of their preference ordering.

[4]The RV and MJ procedures satisfy SCC because these procedures do not aggregate the individual voters' preference orderings into a social preference ordering in order to determine the winner. Under these procedures every candidate is ranked (on a cardinal or ordinal scale) by every voter,

condition (1954, p. 429, postulate 4) which states that if an alternative x chosen from a set T is an element of a subset S of T, then x must be chosen also from S.

3.2.2.7 Preference Inversion Paradox

If the individual preferences of each voter are inverted it is possible that, *ceteris paribus*, the (unique) original winner will still win.

3.2.2.8 Lack of Path Independence Paradox (Farquharson 1969; Plott 1973)

If the voting on the competing candidates is conducted sequentially rather than simultaneously, it is possible that candidate x will be elected under a particular sequence but not, *ceteris paribus*, under an alternative sequence.

3.2.2.9 Strategic Voting Paradox (Gibbard 1973; Satterthwaite 1975)

There are conditions under which a voter with full knowledge of how the other voters are to vote and the decision rule being used, would have an incentive to vote in a manner that does not reflect his true preferences among the competing alternatives. All known non-dictatorial voting procedures suffer from this paradox; it is not demonstrated in the Appendix.

3.3 Voting Procedures for Electing One out of Two or More Candidates

3.3.1 Non-ranked Voting Procedures

There are four main voting procedures for electing a single candidate where voters do not have to rank-order the candidates:

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and the winner is that candidate whose average (or median) rank is highest. Thus the elimination of any losing candidate cannot affect, *ceteris paribus*, the identity of the original winner.

It may perhaps be assumed that under Approval Voting a voter will never vote for an alternative in a subset which s/he did not "approve" in the superset, and hence that Approval Voting, too, satisfies SCC. This assumption is debatable. It can easily be shown – as in Example 3.5.1.1. below – that when there are three alternatives among whom a voter has a linear preference ordering, it would always be rational for a voter under Approval Voting to vote for his/her second preference if his/her top preference is no longer available – even if originally s/he "approved" only of his/her top preference. By doing so s/he has nothing to lose but may obtain a better outcome than by abstaining – regardless of how all other voters are going to vote. Hence in our view Approval Voting may violate SCC.

3.3.1.1 Plurality (or First Past the Post, Briefly FPTP) Voting Procedure

This is the most common procedure for electing a single candidate, and is used, *inter alia*, for electing the members of the House of Commons in the UK and the members of the House of Representatives in the US. Under this procedure every voter casts one vote for a single candidate and the candidate obtaining the largest number of votes is elected.

3.3.1.2 Plurality with Runoff Voting Procedure

Under the usual version of this procedure up to two voting rounds are conducted. In the first round each voter casts one vote for a single candidate. In order to win in the first round a candidate must obtain either a special plurality (usually at least 40% of the votes) or an absolute majority of the votes. If no candidate is declared the winner in the first round then a second round is conducted. In this round only the two candidates who obtained the highest number of votes in the first round participate, and the one who obtains the majority of votes wins. This too is a very common procedure for electing a single candidate and is used, *inter alia*, for electing the President of France.

3.3.1.3 Approval Voting (Brams and Fishburn 1978, 1983)

Under this procedure every voter has a number of votes which is equal to the number of competing candidates, and every voter can cast one vote or no vote for every candidate. The candidate obtaining the largest number of votes is elected. So far this procedure has not been used in any public elections but is already used by several professional associations and universities in electing their officers.

3.3.1.4 Successive Elimination (Farquharson 1969)

This procedure is common in parliaments when voting on alternative versions of bills. According to this procedure voting is conducted in a series of rounds. In each round two alternatives compete; the one obtaining fewer votes is eliminated and the other competes in the next round against one of the alternatives which has not yet been eliminated. The alternative winning in the last round is the ultimate winner.

3.3.2 Ranked Voting Procedures That Are Not Condorcet-Consistent

Six ranked procedures under which every voter must rank-order all competing candidates – but which do not ensure the election of a Condorcet winner when one exists – have been proposed, as far as I know, during the last 250 years. These procedures are described below. Only one of these procedures (alternative vote) is used currently in public elections.

3.3.2.1 Borda's Count (Borda 1784; Black 1958)

This voting procedure was proposed by Jean Charles de Borda in a paper he delivered in 1770 before the French Royal Academy of Sciences entitled 'Memorandum on election by ballot' ('Mémoire sur les élections au scrutin'). According to Borda's procedure each candidate, x, is given a score equal to the number of pairs $(V, \ y)$ where V is a voter and y is a candidate such that V prefers x to y, and the candidate with the largest score is elected. Equivalently, each candidate x gets no points for each voter who ranks x last in his preference ordering, one point for each voter who ranks x second-to-last in his preference order, and so on, and $m - 1$ points for each voter who ranks x first in his preference order (where m is the number of candidates). Thus if all n voters have linear preference orderings among the m candidates then the total number of points obtained by all candidates is equal to the number of voters multiplied by the number of paired comparisons, i.e., to $nm(m - 1)/2$.

3.3.2.2 Alternative Vote (AV); (aka Instant Runoff Voting)

This is the version of the *single transferable vote* (STV) procedure (independently proposed by Carl George Andrae in Denmark in 1855 and by Thomas Hare in England in 1857) for electing a single candidate. It works as follows. In the first step one verifies whether there exists a candidate who is ranked first by an absolute majority of the voters. If such a candidate exists s/he is declared the winner. If no such candidate exists then, in the second step, the candidate who is ranked first by the smallest number of voters is deleted from all ballots and thereafter one again verifies whether there is now a candidate who is ranked first by an absolute majority of the voters. The elimination process continues in this way until a candidate who is ranked first by an absolute majority of the voters is found. The Alternative Vote procedure is used in electing the president of the Republic of Ireland, the Australian House of Representatives, as well as the mayors in some municipal elections in the US.

3.3.2.3 Coombs' Method (Coombs 1964, pp. 397–399; Straffin 1980; Coombs et al. 1984)

This procedure was proposed by the psychologist Clyde H. Coombs in 1964. It is similar to Alternative Vote except that the elimination in a given round under Coombs' method involves the candidate who is ranked last by the largest number of voters (instead of the candidate who is ranked first by the smallest number of voters under alternative vote).

3.3.2.4 Bucklin's Method (Hoag and Hallett 1926, pp. 485–491; Tideman 2006, p. 203)

This voting system can be used for single-member and multi-member districts. It is named after James W. Bucklin of Grand Junction, Colorado, who first promoted it in 1909. In 1913 the US Congress prescribed (in the Federal Reserve Act of 1913, Sect. 4) that this method be used for electing district directors of each Federal Reserve Bank.

Under Bucklin's method voters rank-order the competing candidates. The vote count starts like in the Alternative Vote method. If there exists a candidate who is ranked first by an absolute majority of the voters s/he is elected. Otherwise the number of voters who ranked every candidate in second place are added to the number of voters who ranked him/her first, and if now there exists a candidate supported by a majority of voters s/he is elected. If not, the counting process continues in this way by adding for each candidate his/her third, fourth, ... rankings, until a candidate is found who is supported by an absolute majority of the voters. If two or more candidates are found to be supported by a majority of voters in the same counting round then the one supported by the largest majority is elected.[5]

3.3.2.5 Range Voting (Smith 2000)

According to this procedure the suitability (or level of performance) of every candidate is assessed by every voter and is assigned a (cardinal) grade (chosen from a pre-specified range) reflecting the candidate's suitability or level of performance in the eyes of the voter. The candidate with the highest average grade is the winner. This procedure is currently championed by Warren D. Smith (see http://rangevoting.org) and used to elect the winner in various sport competitions.

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[5]However, it is unclear how a tie between two candidates, say a and b, ought to be broken under Bucklin's procedure when both a and b are supported in the same counting round by the same number of voters and this number constitutes a majority of the voters. If one tries to break the tie between a and b in such an eventuality by performing the next counting round in which all other candidates are also allowed to participate, then it is possible that the number of (cumulated) votes of another candidate, c, will exceed that of a and b.

To see this, consider the following simple example. Suppose there are 18 voters who must elect one candidate under Bucklin's procedure and whose preference orderings among four candidates, a, b, c, d are as follows: seven voters with preference ordering $a \succ b \succ c \succ d$, eight voters with preference ordering $b \succ a \succ c \succ d$, one voter with preference ordering $d \succ c \succ a \succ b$, and two voters with preference ordering $d \succ c \succ b \succ a$. None of the candidates constitutes the top preference of a majority of the voters. However, both a and b constitute the top or second preference by a majority of voters (15). If one tries to break the tie between a and b by performing the next (third) counting round in which c and d are also allowed to participate, then c will be elected (with 18 votes), but if only a and b are allowed to participate in this counting round then b will be elected (with 17 votes).

So which candidate ought to be elected in this example under Bucklin's procedure? As far as I know, Bucklin did not supply an answer to this question.

3.3.2.6 Majority Judgment (Balinski and Laraki 2007a,b, 2011)

According to this proposed procedure, the suitability (or level of performance) of every candidate is assessed by every voter and is assigned an ordinal grade (chosen from a pre-specified range) reflecting the candidate's suitability or level of performance in the eyes of the voter. The candidate with the highest median grade is the winner.

3.3.3 Ranked Voting Procedures that are Condorcet-consistent[6]

All the eight voting procedures described in this subsection require that voters rank-order all competing candidates. Under all these procedures a *Condorcet winner*, if one exists, is elected. The procedures differ from one another regarding which candidate gets elected when the social preference ordering contains a top cycle, i.e., when a Condorcet winner does not exist.

3.3.3.1 The Minimax Procedure

Condorcet specified that the Condorcet winner (whom he called 'the majority candidate') ought to be elected if one exists. However, according to Black (1958, pp. 174–175, 187) Condorcet did not specify clearly which candidate ought to be elected when the social preference ordering contains a top cycle. Black (1958, p. 175) suggests that "It would be most in accordance with the spirit of Condorcet's ... analysis ... to discard all candidates except those with the minimum number of majorities against them and then to deem the largest size of minority to be a majority, and so on, until one candidate had only actual or deemed majorities against each of the others." In other words, the procedure attributed by Black to Condorcet when cycles exist in the social preference ordering is a *minimax procedure*[7] since it chooses that candidate whose worst loss in the paired comparisons is the least bad. This procedure is also known in the literature as the *Simpson–Kramer rule* (see Simpson 1969; Kramer 1977).

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[6] I list here only deterministic procedures. For a Condorcet-consistent probabilistic procedure see Felsenthal and Machover (1992). I also do not list here two Condorcet-consistent deterministic procedures proposed by Tideman (1987) and by Schultze (2003) because I do not consider satisfying (or violating) the independence-of-clones property, which is the main reason why these two procedures were proposed, to be associated with any voting paradox. (A phenomenon where candidate x is more likely to be elected when two clone candidates, y and y', exist, and where x is less likely to be elected when, *ceteris paribus*, one of the clone candidates withdraws, does not seem to me surprising or counter-intuitive).

[7] Young (1977, p. 349) prefers to call this procedure "The minimax function".

3.3.3.2 Dodgson's procedure (Black 1958, pp. 222–234; McLean and Urken, 1995, pp. 288–297)

This procedure is named after the Rev. Charles Lutwidge Dodgson, aka Lewis Carroll, who proposed it in 1876. It elects the Condorcet winner when one exists. If no Condorcet winner exists it elects that candidate who requires the fewest number of switches (i.e. inversions of two adjacent candidates) in the voters' preference orderings in order to make him the Condorcet winner.

3.3.3.3 Nanson's Method (Nanson 1883; McLean and Urken, 1995, ch. 14)

Nanson's method is a recursive elimination of Borda's method. In the first step one calculates for each candidate his Borda score. In the second step the candidates whose Borda score do not exceed the average Borda score of the candidates in the first step are eliminated from all ballots and a revised Borda score is computed for the uneliminated candidates. The elimination process is continued in this way until one candidate is left. If a (strong) Condorcet winner exists then Nanson's method elects him.[8]

3.3.3.4 Copeland's Method (Copeland 1951)

Every candidate x gets one point for every paired comparison with another candidate y in which an absolute majority of the voters prefer x to y, and half a point for every paired comparison in which the number of voters preferring x to y is equal to the number of voters preferring y to x. The candidate obtaining the largest sum of points is the winner.

3.3.3.5 Black's Method (Black 1958, p. 66)

According to this method one first performs all paired comparisons to verify whether a Condorcet winner exists. If such a winner exists then s/he is elected. Otherwise the winner according to Borda's count (see above) is elected.

[8]Although Nanson's procedure satisfies the strong Condorcet condition, i.e., it always elects a candidate who beats every other candidate in paired comparisons, this procedure may not satisfy the weak Condorcet condition which requires that if there exist(s) candidate(s) who is (are) unbeaten by any other candidate then this (these) candidate(s) – and only this (these) candidate(s) – ought to be elected. For an example of violation of the weak Condorcet condition by Nanson's procedure see Niou (1987).

3.3.3.6 Kemeny's Method (Kemeny 1959; Kemeny and Snell 1960; Young and Levenglick 1978; Young 1995)

Kemeny's method (aka *Kemeny–Young rule*) specifies that up to $m!$ possible social preference orderings should be examined (where m is the number of candidates) in order to determine which of these is the "most likely" true social preference ordering.[9] The selected "most likely" social preference ordering according to this method is the one where the number of pairs $(A, \ y)$, where A is a voter and y is a candidate such that A prefers x to y, and y is ranked below x in the social preference ordering is maximized. Given the voters' various preference orderings, Kemeny's procedure can also be viewed as finding the most likely (or the best predictor, or the best compromise) true social preference ordering, called the *median preference ordering*, i.e., that social preference ordering S that minimizes the sum, over all voters i, of the number of pairs of candidates that are ordered oppositely by S and by the ith voter.[10]

3.3.3.7 Schwartz's Method (Schwartz 1972; 1986)

Thomas Schwartz's method is based on the notion that a candidate x deserves to be listed ahead of another candidate y in the social preference ordering if and only if x beats or ties with some candidate that beats y, and x beats or ties with all candidates that y beats or ties with. The Schwartz set (from which the winner should be chosen) is the smallest set of candidates who are unbeatable by candidates outside the set. The Schwartz set is also called *GOCHA* (*Generalized Optimal Choice Axiom*).

3.3.3.8 Young's Method (Young 1977)

According to Fishburn's (1977, p. 473) informal description of Young's procedure "[it] is like Dodgson's in the sense that it is based on altered profiles that have candidates who lose to no other candidate under simple majority. But unlike

[9]Tideman (2006, pp. 187–189) proposes two heuristic procedures that simplify the need to examine all $m!$ preference orderings.

[10]According to Kemeny (1959) the distance between two preference orderings, R and R', is the number of pairs of candidates (alternatives) on which they differ. For example, if $R = a \succ b \succ c \succ d$ and $R' = d \succ a \succ b \succ c$, then the distance between R and R' is 3, because they agree on three pairs $[(a \succ b),\ (a \succ c),\ (b \succ c)]$ but differ on the remaining three pairs, i.e., on the preference ordering between a and d, b and d, and between c and d. Similarly, if R'' is $c \succ d \succ a \succ b$ then the distance between R and R'' is 4 and the distance between R' and R'' is 3. According to Kemeny's procedure the most likely social preference ordering is that R such that the sum of distances of the voters' preference orderings from R is minimized. Because this R has the properties of the median central measure in statistics it is called the *median preference ordering*. The median preference ordering (but not the *mean preference ordering* which is that R which minimizes the sum of the squared differences between R and the voters' preference orderings) will be identical to the possible social preference ordering W which maximizes the sum of voters that agree with all paired comparisons implied by W.

Dodgson, Young deletes voters rather than inverting preferences to obtain the altered profiles. His procedure suggests that we remain most faithful to Condorcet's principle if the choice set consists of alternatives that can become simple majority nonlosers with removal of the fewest number of voters."

3.4 Summary

As can be seen from Tables 3.1–3.3, seven procedures (Alternative Vote, Coombs, Bucklin, Majority Judgment, Minimax, Dodgson, and Young) are susceptible to the largest number of paradoxes (10), whereas the plurality (first-past-the-post) and Borda's procedures are susceptible to the smallest number of paradoxes (6).

Of the nine Condorcet-consistent procedures, six procedures (successive elimination, minimax, Dodgson's, Nanson's, Schwartz's, and Young's) are dominated by the other three procedures (Black's, Copeland's and Kemeny's) in terms of the paradoxes to which these procedures are susceptible.

However, the number of paradoxes to which each of the various voting procedures surveyed here is vulnerable may be regarded as meaningless or even misleading. This is so for two reasons.

Table 3.1 Susceptibility of non-ranked procedures to voting paradoxes

Procedure	Plurality	Plurality w runoff	Approval voting	Successive elimination
Paradox				
Condorcet pdx (cyclical majorities)	−	−	−	+
Condorcet winner pdx	+	+	+	−
Absolute majority pdx	−	−	⊕	−
Condorcet loser pdx	⊕	−	+	−
Absolute loser pdx	⊕	−	⊕	−
Pareto dominated candidate	−	−	⊕	⊕
Lack of monotonicity	−	⊕	−	−
Reinforcement	−	+	−	+
No-show	−	+	−	+
Twin	−	+	−	+
Truncation	−	−	−	+
Subset choice condition (SCC)	+	+	+	+
Preference inversion	+	+	+	−
Path independence	−	−	−	+
Strategic voting	+	+	+	+
Total ⊕ signs	2	1	4	1
Total + & ⊕ signs	6	8	8	9

Notes:

A + sign indicates that a procedure is vulnerable to the specified paradox

A ⊕ sign indicates that a procedure is vulnerable to the specified paradox which seems to us an especially intolerable paradox

A − sign indicates that a procedure is not vulnerable to the specified paradox

It is assumed that all voters have linear preference ordering among all competing candidates

Table 3.2 Susceptibility of ranked non Condorcet-consistent procedures to voting paradoxes

Procedure	Borda	Alternative Vote (AV) STV	Coombs	Bucklin	Range Voting	Majority Judgment
Paradox						
Condorcet pdx (cyclical majorities)	+	+	+	+	+	+
Condorcet winner pdx	+	+	+	+	+	+
Absolute majority pdx	⊕	–	–	–	⊕	⊕
Condorcet loser pdx	–	–	–	⊕	⊕	⊕
Absolute loser pdx	–	–	–	–	⊕	⊕
Pareto dominated candidate	–	–	–	–	–	–
Lack of monotonicity	–	⊕	⊕	–	–	–
Reinforcement	–	+	+	+	–	+
No-show	–	+	+	+	–	+
Twin	–	+	+	+	–	+
Truncation	+	+	+	+	+	+
Subset choice condition (SCC)	+	+	+	+	–	–
Preference inversion	–	+	+	+	–	–
Path independence	–	–	–	–	–	–
Strategic voting	+	+	+	+	+	+
Total ⊕ signs	1	1	1	1	3	3
Total + and ⊕ signs	6	10	10	10	7	10

Notes:
A + sign indicates that a procedure is vulnerable to the specified paradox
A ⊕ sign indicates that a procedure is vulnerable to the specified paradox which seems to us an especially intolerable paradox
A – sign indicates that a procedure is not vulnerable to the specified paradox
It is assumed that all voters have linear preference ordering among all competing candidates

First, some paradoxes are but special cases of other paradoxes or may induce the occurrence of other paradoxes, as follows:

- A procedure which is vulnerable to the absolute majority paradox is also vulnerable to the Condorcet winner paradox;
- A procedure which is vulnerable to the absolute loser paradox is also vulnerable to the Condorcet loser paradox;
- Except for the range voting and majority judgment procedures, all procedures surveyed in this chapter that are vulnerable to the Condorcet loser paradox are also vulnerable to the preference inversion paradox.
- The five procedures surveyed in this chapter which may display lack of monotonicity are also susceptible to the No-Show paradox[11];

[11]Campbell and Kelly (2002) devised a non-monotonic voting rule that does not exhibit the No-Show paradox. However, as this method violates the anonymity and neutrality conditions and hence has not been considered seriously for actual use, we ignore it.

Table 3.3 Susceptibility of ranked Condorcet-consistent procedures to voting paradoxes

Procedure	Minimax	Dodgson	Black	Copeland	Kemeny	Nanson	Schwartz	Young
Paradox								
Condorcet pdx (cyclical majorities)	+	+	+	+	+	+	+	+
Condorcet winner pdx	–	–	–	–	–	–	–	–
Absolute majority pdx	–	–	–	–	–	–	–	–
Condorcet loser pdx	⊕	⊕	–	–	–	–	–	⊕
Absolute loser pdx	⊕	–	–	–	–	–	–	⊕
Pareto dominated cand.	–	–	–	–	–	–	⊕	–
Lack of monotonicity	–	⊕	–	–	–	⊕	–	–
Reinforcement	+	+	+	+	+	+	+	+
No-show	+	+	+	+	+	+	+	+
Twin	+	+	+	+	+	+	+	+
Truncation	+	+	+	+	+	+	+	+
SCC	+	+	+	+	+	+	+	+
Preference inversion	+	+	–	–	–	–	–	+
Path independence	–	–	–	–	–	–	–	–
Strategic voting	+	+	+	+	+	+	+	+
Total ⊕ signs	2	2	0	0	0	1	1	2
Total + & ⊕ signs	10	10	7	7	7	8	8	10

Notes:
A + sign indicates that a procedure is vulnerable to the specified paradox
A ⊕ sign indicates that a procedure is vulnerable to the specified paradox which seems to us an especially intolerable paradox
A – sign indicates that a procedure is not vulnerable to the specified paradox
It is assumed that all voters have linear preference ordering among all competing candidates

- All Condorcet-consistent procedures are susceptible to the no-show paradox and hence also to the twin paradox when there exist at least four candidates.[12]

Second, and more importantly, not all the surveyed paradoxes are equally undesirable. Although assessing the severity of the various paradoxes is largely a subjective matter, *there seems to be a wide consensus that a voting procedure which is susceptible to an especially serious paradox (denoted by ⊕ in Tables 3.1–3.3), i.e., a voting procedure which may elect a pareto-dominated candidate, or elect a Condorcet (and absolute) loser, or display lack of monotonicity, or not elect an absolute winner, should be disqualified as a reasonable voting procedure regardless of the probability that these paradoxes may occur*. On the other hand, the degree of severity that should be assigned to the remaining paradoxes should depend, *inter alia*, on the likelihood of their occurrence under the procedures that are vulnerable

[12]Although all Condorcet-consistent procedures are also susceptible to the Reinforcement paradox, there is no logical connection between this paradox and the no-show paradox. As mentioned by Moulin (1988b, pp. 54–55), when there are no more than three candidates there exist Condorcet-consistent procedures which are immune to both the no-show and twin paradoxes, e.g., the minimax procedure which elects the candidate to whom the smallest majority objects.

to them. Thus, for example, a procedure which may display a given paradox only when the social preference ordering is cyclical – as is the case for most of the paradoxes afflicting the Condorcet-consistent procedures – should be deemed more desirable (and the paradoxes it may display more tolerable) than a procedure which can display the same paradox when a Condorcet winner exists.[13]

Additional criteria which should be used in assessing the relative desirability of a voting procedure are what may be called *administrative-technical criteria*. The main criteria belonging to this category are the following:

- *Requirements from the voter*: some voting procedures make it more difficult for the voter to participate in an election by requiring him/her to rank-order all competing candidates, whereas other procedures make it easier for the voter by requiring him/her to vote for just one candidate or for any candidate(s) s/he approves.
- *Ease of understanding how the winner is selected*: In order to encourage voters to participate in an election a voting procedure must be transparent, i.e., voters must understand how their votes (preferences) are aggregated into a social choice. Thus a voting procedure where the winner is the candidate who received the plurality of votes is easier to explain – and considered more transparent – than a procedure which may involve considerable mathematical calculations (e.g., Kemeny's) in order to determine the winner.
- *Ease of executing the elections*: Election procedures requiring only one voting (or counting) round are more easily executed than election procedures that may require more than one voting (or counting) round. Similarly, election procedures requiring to count only the number of votes received by each candidate are easier to conduct than those requiring the conduct of all $m(m-1)/2$ paired contests between all m candidates, or those requiring the examination of up to $m!$ possible social preference orderings in order to determine the winner.
- *Minimization of the temptation to vote insincerely*: Although all voting procedures are vulnerable to manipulation, i.e., to the phenomenon where some voters may benefit if they vote insincerely, some voting procedures (e.g., Borda's count, Range voting) are susceptible to this considerably more than others.
- *Discriminability*: One should prefer a voting procedure which is more discriminate, i.e., it is more likely to select (deterministically) a unique winner than produce a set of tied candidates – in which case the employment of additional means are needed to obtain a unique winner. Thus, for example, when the social preference ordering is cyclical then, *ceteris paribus*, Schwartz's and Copeland's methods are considerably less discriminating than the remaining Condorcet-consistent procedures surveyed in this chapter.

[13]However, in order to be able to state conclusively which of several voting procedures that are susceptible to the same paradox is more likely to display this paradox, one must know what are the necessary and/or sufficient conditions for this paradox to occur under the various compared procedures. Such knowledge is still lacking with respect to most voting procedures and paradoxes.

Of course there may exist conflicts between some of these technical-administrative criteria. For example, a procedure like Kemeny's which, on the one hand, is more difficult to execute in practice and to explain to prospective voters (and hence less transparent), is, on the other hand, more discriminate and less vulnerable to insincere behavior.

So in view of all the above criteria, which of the 18 surveyed voting procedures do I think should be preferred? Since the weakest extension of the majority rule principle when there are more than two candidates is the Condorcet winner principle, I think that the electoral system which ought to be used for electing one out of $m \geq 2$ candidates should be Condorcet-consistent.

But as one does not know before an election is conducted whether a Condorcet winner will exist or whether the social preference ordering will contain a top cycle, which of the nine Condorcet-consistent procedures surveyed and exemplified in this paper should be preferred in case a top cycle exists? In this case I think that the Successive Elimination procedure and Schwartz's procedure should be readily disqualified because of their vulnerability to electing a pareto-dominated candidate, Dodgson's and Nanson's procedures should be readily disqualified because of their lack of monotonicity, and the minimax and Young's procedures should be readily disqualified because of their vulnerability to electing an absolute or a Condorcet loser. Although Black's procedure cannot elect a Condorcet loser, it may nevertheless come quite close to it because, as demonstrated in Example 3.5.13.3 below, it violates Smith's (1973) Condorcet principle, so this procedure too seems to me not considerably more desirable than the minimax and Young's procedures.

This leaves us with a choice between the remaining two Condorcet-consistent procedures – Copeland's and Kemeny's. The choice between them depends on the importance one assigns to the above-mentioned technical-administrative criteria. Both these procedures require voters to rank-order all candidates. However, Copeland's method is probably easier than Kemeny's to explain to lay voters, as well as, when the number of candidates is large, may involve considerably fewer calculations in determining who is (are) the ultimate winner(s). Kemeny's procedure, on the other hand, is more discriminate than Copeland's when the number of candidates is relatively small, and is probably also – because of its increased complexity in determining the ultimate winner – less vulnerable to insincere voting. So if I would have to choose between these two procedures I would choose Kemeny's because most elections where a single candidate must be elected usually involve relatively few contestants – in which case Kemeny's procedure seems to have an advantage over Copeland's procedure. Moreover, as I mentioned in the description of Kemeny's procedure and as argued by Young (1995, pp. 60–62), Kemeny's procedure has also the advantage that it can be justified not only from Condorcet's perspective of the maximum likelihood rule, but also as choosing for the entire society the "median preference ordering" – which can be viewed from the perspective of modern statistics as the best compromise between the various rankings reported by the voters.

3.5 Appendix: exemplifying the Various Paradoxes That Afflict the Various Procedures

3.5.1 Demonstrating Paradoxes Afflicting the Plurality Procedure

Except for being vulnerable to strategic voting, the plurality procedure is vulnerable to the Condorcet winner paradox, the Condorcet loser paradox, the absolute loser paradox, the preference inversion paradox, and to SCC. The following example demonstrates the vulnerability of the plurality procedure to all these paradoxes simultaneously.

3.5.1.1 Example

Suppose there are nine voters who must elect one out of three candidates, a, b, and c, and whose preference orderings among these candidates are as follows:

No. of voters	Preference ordering
4	$a \succ b \succ c$
3	$b \succ c \succ a$
2	$c \succ b \succ a$

Here b is the Condorcet winner and a is not only a Condorcet loser but also an absolute loser. Nevertheless, if all voters vote for their top preference then a will be elected. Note that if c drops out of the race then b will be elected – thus demonstrating violation of SCC. Note also that if all voters invert their preference orderings then a becomes an absolute winner and hence will be elected – thus demonstrating the Preference Inversion paradox.

3.5.2 Demonstrating Paradoxes Afflicting the Plurality with Runoff Procedure

Except for being vulnerable to strategic voting, the plurality with runoff procedure is vulnerable to the Condorcet winner, lack of monotonicity, reinforcement, no-show, twin, preference inversion, and to the SCC paradoxes.

Example 3.5.2.1 below demonstrates the vulnerability of the plurality with runoff procedure to the Condorcet winner, to lack of monotonicity, and to the SCC paradoxes.

3.5.2.1 Example

Suppose there are 43 voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
7	$a \succ b \succ c$
9	$a \succ c \succ b$
14	$b \succ c \succ a$
13	$c \succ a \succ b$

Here the social preference ordering is $c \succ a \succ b$, i.e., c is the Condorcet winner. But if all voters vote sincerely then under the plurality with runoff procedure c will be eliminated in the first round and a will beat b in the second round and thus become the ultimate winner. (Note that if c would have withdrawn from the race prior to the first round then, *ceteris paribus*, a would have been elected already in the first round, thereby demonstrating this procedure's vulnerability to SCC).

Now suppose that, *ceteris paribus*, five of the 14 voters whose preference ordering is $b \succ c \succ a$ (who are not very happy with the prospect that a may be elected) change it to $a \succ b \succ c$ thereby increasing a's support. As a result of this change b (rather than c) will be eliminated in the first round, and c (the Condorcet winner) will beat a in the second round – thereby demonstrating the vulnerability of the plurality with runoff procedure to non-monotonicity.

Example 3.5.2.2 demonstrates the vulnerability of the plurality with runoff procedure to the reinforcement paradox.

3.5.2.2 Example

Suppose there are two districts, I and II. In district I there are 17 voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
4	$a \succ b \succ c$
1	$b \succ a \succ c$
5	$b \succ c \succ a$
6	$c \succ a \succ b$
1	$c \succ b \succ a$

and in district II there are 15 voters whose preference orderings among the three candidates are as follows:

No. of voters	Preference ordering
6	$a \succ c \succ b$
8	$b \succ c \succ a$
1	$c \succ a \succ b$

If all voters vote sincerely then no candidate is ranked first by an absolute majority of the voters in district I. Consequently candidate a is deleted from the race after the first round and candidate b beats candidate c in this district in the second round.

In district II candidate b, who is ranked first by the majority of voters, is elected in the first round.

However if, *ceteris paribus*, the two districts are amalgamated into a single district, we obtain the following distribution of preference orderings of the 32 voters:

No. of voters	Preference ordering
4	$a \succ b \succ c$
6	$a \succ c \succ b$
1	$b \succ a \succ c$
13	$b \succ c \succ a$
7	$c \succ a \succ b$
1	$c \succ b \succ a$

If all voters vote sincerely then no candidate is ranked first by an absolute majority of the voters. Consequently c is deleted after the first round and a beats b and is elected in the second round – in violation of the reinforcement postulate.

Example 3.5.2.3 demonstrates the vulnerability of the plurality with runoff procedure to the no-show and to the twin paradoxes.

3.5.2.3 Example

Suppose there are 11 voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
4	$a \succ b \succ c$
3	$b \succ c \succ a$
1	$c \succ a \succ b$
3	$c \succ b \succ a$

If all voters vote sincerely then no candidate is ranked first by an absolute majority of the voters. Consequently b is deleted after the first round and c beats a in the second round and is elected. Since the election of c is the worst outcome for the voters whose preference ordering is $a \succ b \succ c$, suppose that, *ceteris paribus*, two of them decide not to participate in the election (no-show). We thus obtain the following distribution of preference orderings:

No. of voters	Preference ordering
2	$a \succ b \succ c$
3	$b \succ c \succ a$
1	$c \succ a \succ b$
3	$c \succ b \succ a$

Here a (rather than b) is eliminated in the first round, and b beats c in the second round. Thus the $a \succ b \succ c$ voters obtained, *ceteris paribus*, a better outcome when two of them did not participate in the election than when all of them participated in the election thereby demonstrating the no-show paradox.

This example demonstrates also the vulnerability of the plurality with runoff procedure to the (weak form) of the twin paradox. Suppose that, *ceteris paribus*, there are only two voters with preference ordering $a \succ b \succ c$. One would expect these voters to welcome another "twin" voters having identical preference ordering to theirs thereby presumably giving an increased weight to their common preference ordering. Yet as we saw, the addition of these twins to the electorate results in the election of c, their worst alternative – thereby demonstrating the twin paradox.

Example 3.5.2.4 demonstrates the vulnerability of the plurality with runoff procedure to the preference inversion paradox.

3.5.2.4 Example

Suppose there are 11 voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
5	$a \succ b \succ c$
4	$b \succ c \succ a$
2	$c \succ a \succ b$

If all voters vote sincerely for their top preference in the first round, then c will be eliminated at the end of the first round and thereafter a will beat b in the second round. However, if all voters invert their preference orderings then b will be eliminated at the end of the first round and a will beat c in the second round – thus demonstrating the Preference inversion paradox.

3.5.3 *Demonstrating the Paradoxes Afflicting the Approval Voting Procedure*

Except for being vulnerable to strategic voting, the approval voting procedure is vulnerable to the Condorcet winner paradox, the Condorcet loser paradox, the absolute majority and absolute loser paradoxes, to the pareto-dominated paradox, to the Preference Inversion paradox, and to SCC.

Example 3.5.3.1 demonstrates the vulnerability of the approval voting procedure to the Condorcet winner paradox.

3.5.3.1 Example

This example is due to Felsenthal and Maoz (1988, p. 123, Example 3.2). Suppose there are 47 voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
18	$(a) \succ b \succ c$
6	$(b \succ c) \succ a$
8	$(b \succ a) \succ c$
2	$(c \succ a) \succ b$
13	$(c) \succ b \succ a$

The social preference ordering is $b \succ a \succ c$, i.e., b is the Condorcet winner. However, if all voters approve (and vote for) the candidates denoted between parentheses then a would get the largest number of approval votes (28) and will thus be elected.

Example 3.5.3.2 demonstrates the vulnerability of the approval voting procedure to the pareto-dominated paradox.

3.5.3.2 Example

This example is due to Felsenthal and Maoz (1988, p. 123, Example 3.4). Suppose there are three voters whose preference orderings among four candidates, a, b, c, and d, are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c \succ d$
1	$c \succ a \succ b \succ d$
1	$d \succ a \succ b \succ c$

The social preference ordering is $a \succ b \succ c \succ d$, i.e., a is the Condorcet winner. However, if each voter approves (and votes for) his top three preferences then a tie would occur between the number of votes (3) obtained by candidates a and b, and if this tie were to be broken randomly then there is a 0.5 probability that b would be elected. So if b were to be elected it would demonstrate not only that the Condorcet winner (a) was not elected but also that a pareto-dominated candidate can be elected under the approval voting procedure. (Note that *all* voters prefer a to b).

Example 3.5.3.3 demonstrates the vulnerability of the approval voting procedure to the absolute majority paradox.

3.5.3.3 Example

Suppose there are 100 voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
51	$a \succ b \succ c$
48	$b \succ c \succ a$
1	$c \succ b \succ a$

The social preference ordering is $a \succ b \succ c$, i.e., a is the Condorcet winner who is ranked first by an absolute majority of the voters. However, if only one candidate must be elected and if each voter approves (and votes for) his top two preferences, then b will be elected despite the fact that a is ranked first by an absolute majority of the voters.

Example 3.5.3.4 demonstrates the vulnerability of the approval voting procedure to the absolute loser and to the Condorcet loser paradoxes.

3.5.3.4 Example

Suppose there are 15 voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
6	$(a) \succ b \succ c$
4	$(b) \succ c \succ a$
1	$(c \succ a) \succ b$
4	$(c) \succ b \succ a$

The social preference ordering is $b \succ c \succ a$, i.e., a is not only the Condorcet loser but also the Absolute Loser because this candidate is ranked last by an absolute majority of the voters. However, if only one candidate must be elected and if all voters approve (and vote for) the candidate(s) denoted between parentheses then a will be elected.

This example can also be used to demonstrate the susceptibility of the Approval Voting procedure to the preference inversion paradox. If in the above example all voters invert their preference ordering and decide to vote, as before, either only for their top preference or for their top two preferences, then we obtain the following distribution of votes:

No. of voters	Preference ordering
6	$(c) \succ b \succ a$
4	$(a) \succ c \succ b$
1	$(b \succ a) \succ c$
4	$(a) \succ b \succ c$

Here a is not only the Condorcet winner but also the absolute winner and is elected – thereby demonstrating the susceptibility of approval voting to the preference inversion paradox.

When all voters are assumed to approve of (and vote for) originally only their top preference (as under the plurality procedure) – and subject to what we said in footnote 4 above – Example 3.5.1.1 can be used to also demonstrate the susceptibility of the Approval Voting procedure to SCC. Thus, for instance, it would be worthwhile, *ceteris paribus*, for the two voters in Example 3.5.1.1 whose original preference ordering is $c \succ b \succ a$ to vote for b if alternative c were no longer available even though they did not "approve" originally of b – because by voting for b they lose nothing but may avert the election of a, their least preferable alternative, which may be elected if they abstain.

3.5.4 Demonstrating the Paradoxes Afflicting the Successive Elimination Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, the successive elimination procedure is vulnerable to pareto-dominated, reinforcement, no-show, twin, truncation, SCC, and path independence paradoxes.

Example 3.5.4.1 demonstrates the vulnerability of the successive elimination procedure to the election of a pareto-dominated candidate. A necessary condition for this to happen is that the social preference ordering is cyclical and there are at least four candidates (Fishburn, 1982, p. 131).

3.5.4.1 Example

Suppose there are 11 voters whose preference orderings among four candidates, a, b, c, and d, are as follows:

No. of voters	Preference ordering
3	$a \succ b \succ c \succ d$
2	$c \succ a \succ b \succ d$
1	$c \succ d \succ a \succ b$
5	$d \succ a \succ b \succ c$

Thus the social preference ordering is cyclical ($b \succ c \succ d \succ a \succ b$). Suppose further that all the voters always vote sincerely for their preferred candidate in each round, and that the order in which the divisions are carried out is as follows:

In round 1: d against a;
In round 2: the winner of round 1 against c;
In round 3: the winner of round 2 against b;

Given this order d beats a (6:5) in the first round, c beats d (6:5) in the second round, and b beats c (8:3) in the third round and becomes the ultimate winner. Note, however, that b is a Pareto-dominated candidate because *all* the voters prefer a to b.

This example can also be used to demonstrate the vulnerability of the successive elimination procedure to SCC.

If, *ceteris paribus*, d is deleted, then in the first round a will beat c (8:3), and in the second round a will beat b (11:0) and thus a will become the ultimate winner – in violation of SCC.

Similarly, this example can also be used to demonstrate the vulnerability of the successive elimination procedure to the no-show paradox.

If, *ceteris paribus*, two of the voters whose top preference is d decide not to participate, then a becomes the Condorcet winner and hence will be elected under the successive elimination procedure. Note that this outcome is preferred over the election of b by the two $d \succ a \succ b \succ c$ voters who decided not to participate – thus demonstrating the vulnerability of the successive elimination procedure to the no-show paradox.

This example can also be used to demonstrate the vulnerability of the successive elimination procedure to lack of path independence when the social preference ordering is cyclical.

Given the above preference orderings of the 11 voters, if the order of the divisions in each round were changed such that:

In round 1: a against b
In round 2: the winner of round 1 against c
In round 3: the winner of round 2 against d

Then in the first round a would beat b (11:0), in the second round a would also beat c (8:3), but in the third round d would beat a (6:5) and become the ultimate winner.

Example 3.5.4.2 demonstrates the vulnerability of the successive elimination procedure to the reinforcement paradox.

3.5.4.2 Example

Suppose there are two districts, I and II. In district I there are three voters whose preference orderings among four candidates are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ d \succ c$
1	$b \succ d \succ c \succ a$
1	$d \succ c \succ a \succ b$

and in district II there are four voters whose preference ordering among the four candidates are as follows:

No. of Voters	Preference Ordering
3	$c \succ b \succ d \succ a$
1	$d \succ a \succ b \succ c$

If the order of divisions in each district is:

b vs. d in round 1
Winner of first round against a in round 2
Winner of second round against c in round 3

Then in each district c will be the ultimate winner.

However if, *ceteris paribus*, the two districts are amalgamated into a single district of seven voters, then d becomes the Condorcet winner and will therefore be elected under the successive elimination procedure – in violation of the reinforcement postulate.

Example 3.5.4.3 demonstrates the vulnerability of the successive elimination procedure to the Twin paradox.

3.5.4.3 Example

This example is due to Moulin (1988b, p. 54). Suppose there are six voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
2	$a \succ b \succ c$
2	$b \succ c \succ a$
1	$c \succ a \succ b$
1	$c \succ b \succ a$

Suppose further that the order in which the divisions are conducted is as follows:

a vs. b in round 1
Winner of round 1 vs. c in round 2

and that if there is a tie between two candidates in any of the divisions it is broken lexicographically, i.e., in favor of the candidate who is denoted by the letter that is closer to the beginning of the alphabet.

Accordingly, there is a tie between a and b in the first round which is broken in favor of a, and in the second round c beats a and becomes the ultimate winner.

In view of this result one could expect that, *ceteris paribus*, the single $c \succ b \succ a$ voter should welcome if an additional "twin" voter would join the electorate thereby providing more weight to their common preferences. However, an addition of a second $c \succ b \succ a$ voter would result, *ceteris paribus*, in a net loss to the first $c \succ b \succ a$ voter because b would become the Condorcet winner and hence also the ultimate winner under the successive elimination procedure – thus demonstrating the twin paradox.

Example 3.5.4.4 demonstrates the vulnerability of the successive elimination procedure to the Truncation paradox.

3.5.4.4 Example

Suppose there are six voters with the following preference orderings:

No. of voters	Preference ordering
1	$a \succ b \succ c \succ d$
1	$c \succ b \succ a \succ d$
2	$c \succ d \succ b \succ a$
2	$d \succ a \succ b \succ c$

Suppose further that the order in which the divisions are conducted is as follows:

First round: b vs. c
Second round: winner of first round vs. d
Third round: winner of second round vs. a

Additionally, suppose that if a tie occurs between two candidates it is broken in favor of the one denoted by a letter closer to the beginning of the alphabet.

Accordingly, in the first round there is a tie between b and c which is broken in favor of b. In the second round d beats b, and in the third round d beats a and hence becomes the ultimate winner. This is of course a very bad outcome for the single voter whose preference ordering is $a \succ b \succ c \succ d$. So suppose that, *ceteris paribus*, this voter would truncate his preferences between b, c, and d, and indicate just his top preference, a, i.e., this voter will participate only in the third round in which a will compete against the winner from the second round. As a result of such truncation c would beat b in the first round, c would beat also d in the second round, but in the third round there would be a tie between a and c – which will be broken in favor of a, a much better result for the $a \succ b \succ c \succ d$ voter, thus demonstrating the truncation paradox.

3.5.5 Demonstrating Paradoxes Afflicting Borda's Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, Borda's procedure is vulnerable to the Condorcet winner, absolute majority, truncation, and SCC paradoxes. And as I shall show in Example 3.5.13.3, it also violates Smith's Condorcet principle.

Example 3.5.5.1 demonstrates simultaneously the vulnerability of Borda's procedure to the absolute majority paradox (and thus also to the Condorcet winner paradox).

3.5.5.1 Example

Suppose there are 100 voters who have to elect one out of three candidates, a, b, c, under Borda's procedure, and whose preference orderings are as follows:

No. of voters	Preference ordering
51	$a \succ b \succ c$
48	$b \succ c \succ a$
1	$c \succ b \succ a$

The number of Borda points awarded to candidates a, b, and c, are 102, 148, and 50, respectively, so candidate b is elected. However, note that candidate a is not only the Condorcet winner but also an absolute winner because an absolute majority of the voters rank candidate a as their top preference.

Example 3.5.5.2 demonstrates the vulnerability of Borda's procedure to the truncation paradox.

3.5.5.2 Example

This example is due to Fishburn (1974, p. 543). Suppose that seven voters have to elect one out of four candidates $a - d$ under Borda's procedure, and that their preference orderings among the candidates are as follows:

No. of voters	Preference ordering
3	$a \succ b \succ c \succ d$
1	$b \succ c \succ a \succ d$
1	$b \succ c \succ d \succ a$
2	$c \succ d \succ a \succ b$

Suppose further that under Borda's procedure with k candidates one assigns k points to the top-ranked candidate, $k-1$ points to the second-ranked candidate, ..., 1 point to the kth ranked candidate, and 0 points to any non-ranked candidate.

Given the above preference orderings and Borda-point assignment, the number of points awarded to candidates a, b, c, and d, are 19, 19, 20, and 12, respectively, so candidate c is elected. However, if the first three voters (who are not very happy with the election of candidate c) decide not to rank (i.e., truncate) candidate c, then the number of Borda points awarded to candidates a, b, c, and d, are 16, 16, 14, and 12, respectively, so candidates a and b are tied and one of them will be eventually elected depending on the rule employed for braking ties. This result is of course preferred by the first three voters to the election of candidate c, thereby demonstrating the truncation paradox.

Example 3.5.5.3 demonstrates the vulnerability of Borda's procedure to SCC.

3.5.5.3 Example

Suppose that 11 voters have to elect one out of three candidates, a, b, or c, under Borda's procedure and that their preference orderings among these candidates are as follows:

No. of voters	Preference ordering
3	$a \succ c \succ b$
3	$b \succ a \succ c$
5	$c \succ b \succ a$

Accordingly, the number of Borda points awarded to candidates *a, b*, and *c*, are 9,11, and 13, respectively – so candidate *c* is elected.

Now suppose that, *ceteris paribus*, candidate *b* drops out of the race. In this case the number of Borda points awarded to candidates *a* and *c* are 6 and 5, respectively, so candidate *a* would be elected – in violation of SCC.

3.5.6 Demonstrating Paradoxes Afflicting the Alternative Vote Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, the Alternative Vote procedure is vulnerable to the Condorcet winner, lack of monotonicity, reinforcement, no-show, twin, truncation, preference inversion, and SCC paradoxes.

The same examples that were used to demonstrate the vulnerability of the plurality with runoff procedure to all these paradoxes (except the truncation paradox), can also be used to demonstrate the vulnerability of the alternative vote procedure to these paradoxes.

Specifically, Example 3.5.2.1 above can be used to demonstrate the vulnerability of the Alternative Vote procedure to the Condorcet winner, to lack of monotonicity,[14] and to the SCC paradoxes; Example 3.5.2.2 above can be used to demonstrate the vulnerability of the Alternative Vote procedure to the reinforcement paradox, Example 3.5.2.3 above can be used to demonstrate the vulnerability of the alternative vote procedure to the no-show and twin paradoxes, and Example 3.5.2.4 above can be used to demonstrate the vulnerability of the alternative vote procedure to the preference inversion paradox.

Example 3.5.6.1 demonstrates the vulnerability of the alternative vote procedure to the truncation paradox.

[14] A display of negative responsiveness (or lack of monotonicity) under the alternative vote procedure has actually occurred recently in the March 2009 mayoral election in Burlington, Vermont. Among the three biggest vote getters, the Republican got the most first-place votes, the Democrat the fewest, and the Progressive won after the Democrat was eliminated. Yet if many of those who ranked the Republican first had ranked the Progressive first, the Republican would have been eliminated and the Progressive would have lost to the Democrat. In March 2010 Burlington replaced the Alternative Vote procedure for electing its mayor with the Plurality with Runoff procedure – which is also susceptible to negative responsiveness. See detailed report in http://rangevoting.org/Burlington.html.

3.5.6.1 Example

This example is due to Nurmi (1999, p. 63). Suppose there are 103 voters whose preference orderings among four candidates, a, b, c, and d, are as indicated below and who must elect one of these candidates under the alternative vote procedure.

No. of voters	Preference ordering
33	$a \succ b \succ c \succ d$
29	$b \succ a \succ c \succ d$
24	$c \succ b \succ a \succ d$
17	$d \succ c \succ b \succ a$

Since none of the four candidates is ranked first by an absolute majority of the voters, candidate d (who is ranked first by the smallest number of voters) is eliminated. As this does not yet lead to a winner, b is eliminated, whereupon a wins.

Suppose now that, *ceteris paribus*, those 17 voters who rank a last decide to truncate their preference ordering and list only their top preference, d. In this case d will be eliminated first (as before), but since these 17 voters did not indicate their preference ordering among the remaining candidates, candidate c (rather than b) will be eliminated thereafter – whereupon b wins. This result is preferred by these 17 voters to the election of a, thereby demonstrating the truncation paradox.

3.5.6.2 Remark

As stated at the outset of this chapter, the UK conducted a referendum in May 2011 regarding whether to replace its plurality voting procedure in parliamentary elections with the alternative vote procedure. It may therefore be interesting to note that when there are only three competing candidates (as is usually the case in parliamentary elections in England), the alternative vote procedure is more Condorcet-efficient than the plurality procedure. This is so because, by definition, a necessary and sufficient condition for a Condorcet winner (or any other candidate) to be elected under the plurality procedure is that s/he will constitute the top preference of a plurality of the voters, whereas for a Condorcet winner to be elected under the alternative vote procedure when there are three candidates it is sufficient (but not necessary) that the Condorcet winner constitutes the top preference of a plurality of the voters. This is so because if there exist three candidates, a, b, and c, such that the social preference ordering is $a \succ b \succ c$ and a constitutes the top preference of the plurality of voters, then either b or c (but not a) must be eliminated in the first counting round, and as a is the Condorcet winner s/he must necessarily beat the remaining alternative in the second counting round.

So while it is a sufficient condition for a Condorcet winner to be elected under the alternative vote procedure when there are three candidates and the Condorcet winner constitutes the top preference of a plurality of the voters, it is not a necessary condition because, as can be ascertained from Example 3.5.1.1, a Condorcet winner can be elected under the Alternative Vote procedure when there are three candidates even though it does not constitute the top preference of a plurality of the voters.

However, it is no longer a sufficient condition for a Condorcet winner who is ranked first by a plurality of the voters to be elected under the alternative vote procedure once there are more than three candidates. This is demonstrated in Example 3.5.6.3.

3.5.6.3 Example

This example is due partly to Moshé Machover who provided me general guidance in its construction (private communication 13.12.2010). Suppose there are 85 voters whose preference orderings among four candidates, a, b, c, and d, are as indicated below and who must elect one of these candidates under the alternative vote procedure.

No. of voters	Preference ordering
15	$a \succ b \succ c \succ d$
10	$a \succ c \succ b \succ d$
13	$b \succ a \succ c \succ d$
10	$b \succ c \succ a \succ d$
14	$c \succ a \succ b \succ d$
10	$c \succ b \succ a \succ d$
6	$d \succ c \succ a \succ b$
7	$d \succ b \succ a \succ c$

The social preference ordering here is $a \succ b \succ c \succ d$, i.e., candidate a is the Condorcet winner who is ranked first by a plurality of the voters. However, as none of the candidates is ranked first by an absolute majority of the voters, one deletes first candidate d according to the alternative vote procedure, and thereafter one deletes candidate a, whereupon candidate b becomes the winner.

3.5.7 Demonstrating Paradoxes Afflicting Coombs' Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, Coombs' procedure is vulnerable to the same paradoxes afflicting the alternative vote procedure, i.e., the Condorcet winner, monotonicity, reinforcement, no-show, twin, truncation, preference inversion, and the SCC paradoxes.

Example 3.5.7.1 demonstrates the vulnerability of Coombs' procedure to the Condorcet winner paradox.

3.5.7.1 Example

This example is due to Nicolaus Tideman (private communication 8.9.2010). Suppose that 45 voters have to elect under Coombs' procedure one out of three candidates, a, b, or c, and that their preference orderings among these three candidates are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c$
10	$a \succ c \succ b$
11	$b \succ a \succ c$
11	$b \succ c \succ a$
10	$c \succ a \succ b$
2	$c \succ b \succ a$

The social preference ordering is $b \succ c \succ a$, i.e., b is the Condorcet winner. However, since none of the candidates is ranked first by an absolute majority of the voters, one deletes according to Coombs' procedure the candidate who is ranked last by the largest number of voters. In the above example this candidate is b, the Condorcet winner. (After deleting b candidate c is ranked first by an absolute majority of the voters and is elected.)

3.5.7.2 Remark

It is not clear whether Coombs' procedure is more Condorcet-efficient than either the plurality or the alternative vote procedures. As we have already proved in Remark 3.5.6.2, a necessary and sufficient condition for a Condorcet winner to be elected under the plurality procedure is that the Condorcet winner constitutes the top preference of a plurality of the voters. This condition is sufficient (but not necessary) for a Condorcet winner to be elected under the alternative vote procedure when there are three candidates. However, as is demonstrated in Example 3.5.7.1 above, this condition is neither necessary nor sufficient for a Condorcet winner to be elected under Coombs' procedure. On the other hand, as argued by Coombs (1964, p. 399), a sufficient condition for a Condorcet winner to be elected under his proposed procedure is that the voters' preferences are single-peaked along s single dimension. But under both the plurality and alternative vote procedures a Condorcet winner may not be elected when the voters' preferences are single-peaked along a single dimension. To see this consider Example 3.5.7.3.

3.5.7.3 Example

Suppose there are 13 voters who must elect one out of three candidates, a, b, or c, and whose preference orderings among these candidates are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c$
2	$a \succ c \succ b$
4	$b \succ a \succ c$
6	$c \succ a \succ b$

Here a is the Condorcet winner, the voters' preferences are single-peaked, and a is elected under Coombs' procedure. However, under the plurality and alternative vote procedures c is elected.

Example 3.5.7.4 demonstrates the vulnerability of Coombs' procedure to non-monotonicity.

3.5.7.4 Example

In Example 3.5.7.1 above candidate c was elected under Coombs' procedure although candidate b is the Condorcet winner. Now suppose that, *ceteris paribus*, the 11 voters whose preference ordering is $b \succ a \succ c$ (who are not happy with the prospect that c will be elected) decide to *increase* c's support by changing their preference ordering to $b \succ c \succ a$. Candidate b is still the Condorcet winner but as a result of this change a (rather than b) will first be eliminated under Coombs' procedure, and thereafter b will be elected – in violation of the monotonicity postulate.

Example 3.5.7.5 demonstrates the vulnerability of Coombs' procedure to the no-show, truncation, and preference inversion paradoxes.

3.5.7.5 Example

Suppose there are 15 voters who must elect one out of three candidates, a, b, or c, under Coombs' procedure, and whose preference orderings among these candidates are as follows:

No. of voters	Preference ordering
4	$a \succ b \succ c$
4	$b \succ c \succ a$
5	$c \succ a \succ b$
2	$c \succ b \succ a$

Here no candidate is ranked first by an absolute majority of the voters. Hence, according to Coombs' procedure, a is eliminated in the first round and thereafter b is elected.

Now suppose that, *ceteris paribus*, the two voters with preference ordering $c \succ b \succ a$ decide not to participate in the election. In this case b is eliminated according to Coombs' procedure in the first round and thereafter c (the abstainers' top preference!) is elected thereby demonstrating the no-show paradox.

This example can also be used to demonstrate the vulnerability of Coombs' procedure to the truncation paradox: if the two voters with preference ordering $c \succ b \succ a$ decide to list only their top preference then, *ceteris paribus*, b would be eliminated according to Coombs' procedure and thereafter c would be elected!

If, *ceteris paribus*, all voters invert their preference orderings, then we obtain the following distribution of votes:

No. of voters	Preference ordering
4	$c \succ b \succ a$
4	$a \succ c \succ b$
5	$b \succ a \succ c$
2	$a \succ b \succ c$

As no candidate obtains an absolute majority of the votes in the first counting round, c is eliminated and thereafter b is elected in the second counting round – thus demonstrating the vulnerability of Coombs' procedure to the preference inversion paradox.

Example 3.5.7.6 demonstrates the vulnerability of Coombs' procedure to the Reinforcement paradox.

3.5.7.6 Example

Suppose there are two districts, I and II. In district I there are 34 voters whose preference orderings among three candidates, a, b, and c, are as follows:

No. of voters	Preference ordering
9	$a \succ b \succ c$
9	$b \succ c \succ a$
11	$c \succ a \succ b$
5	$c \succ b \succ a$

and in district II there are seven voters whose preference orderings among the three candidates are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c$
6	$b \succ a \succ c$

Since no candidate is ranked first by an absolute majority of the voters in district I candidate a is eliminated under Coombs' procedure in the first round, and thereafter candidate b is elected. In district II candidate b is ranked first by an absolute majority of the voters and is elected right away.

However, if, *ceteris paribus*, the two districts are amalgamated into a single district of 41 voters then one obtains the following distribution of preferences:

No. of voters	Preference ordering
10	$a \succ b \succ c$
6	$b \succ a \succ c$
9	$b \succ c \succ a$
11	$c \succ a \succ b$
5	$c \succ b \succ a$

Since none of the three candidates is ranked first in the amalgamated district, candidate c is eliminated according to Coombs' procedure in the first round, and candidate a (rather than b) is elected thereafter – thus demonstrating the reinforcement paradox.

Example 3.5.7.7 demonstrates the vulnerability of Coombs' procedure to the Twin paradox.

3.5.7.7 Example

Suppose there are 20 voters who have to choose one out of four candidates, a,b,c, or d, under Coombs' procedure and whose preference orderings among these candidates are as follows:

No. of voters	Preference ordering
5	$a \succ b \succ d \succ c$
5	$b \succ c \succ d \succ a$
1	$b \succ a \succ d \succ c$
6	$c \succ a \succ d \succ b$
1	$c \succ b \succ a \succ d$
2	$c \succ b \succ d \succ a$

Since no voter is ranked first by an absolute majority of the voters, candidate a is eliminated according to Coombs' procedure in the first round and thereafter b is elected.

Now suppose that, *ceteris paribus*, two more voters with preference ordering $b \succ a \succ d \succ c$ join the electorate thereby apparently increasing the chances of candidate b to be elected. However, as result of this increase of the electorate candidate c (rather than a) will be eliminated in the first round under Coombs' procedure, and thereafter a tie will be created between candidates a and b – thereby *decreasing* the chances of candidate b to be elected if the tie is to be broken randomly.

Example 3.5.7.8 demonstrates the vulnerability of Coombs' procedure to SCC.

3.5.7.8 Example

Suppose that there are 29 voters having to elect under Coombs' procedure one out of four candidates, a, b, c, or d, and whose preference orderings among the four candidates are as follows:

No. of voters	Preference ordering
11	$a \succ b \succ c \succ d$
12	$b \succ c \succ d \succ a$
2	$b \succ a \succ d \succ c$
4	$c \succ a \succ d \succ b$

Since none of the candidates is ranked first by an absolute majority of the voters, one deletes according to Coombs' procedure the candidate who is ranked last by the largest number of voters. In the above example this candidate is a. After deleting a candidate b is ranked first by an absolute majority of the voters and is elected.

Now suppose that, *ceteris paribus*, candidate c drops out of the race. As a result candidate a is ranked first by an absolute majority of the voters and is elected – contrary to SCC.

3.5.8 Demonstrating Paradoxes Afflicting Bucklin's Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, Bucklin's procedure is vulnerable to the Condorcet winner, Condorcet loser, reinforcement, no-show, twin, truncation, preference inversion, and SCC paradoxes.

Example 3.5.7.1 above can be used to demonstrate the susceptibility of Bucklin's procedure to the Condorcet winner paradox. In this example b is the Condorcet winner but under Bucklin's procedure a is elected because the number of voters (32) who rank a first or second exceeds the number of voters (25) who rank b first or second.

Example 3.5.8.1 demonstrates that a Condorcet loser may be elected under Bucklin's procedure.

3.5.8.1 Example

This example is due to Tideman (2006, p. 197, Example 13.13). Suppose there are 29 voters whose preference orderings among four candidates, w, x, y, z, are as follows:

No. of voters	Preference ordering
5	$w \succ x \succ y \succ z$
3	$w \succ z \succ x \succ y$
5	$x \succ y \succ w \succ z$
2	$x \succ z \succ y \succ w$
3	$y \succ w \succ x \succ z$
2	$y \succ z \succ w \succ x$
4	$z \succ w \succ x \succ y$
2	$z \succ x \succ y \succ w$
3	$z \succ y \succ w \succ x$

The social preference ordering here contains a top cycle ($w \succ x \succ y \succ w$), but since each of the three candidates w, x, y beats candidate z in pairwise contests, candidate z is a Condorcet loser. However, under Bucklin's procedure candidate z will be elected because the number of voters (16) who rank z first or second in their preference ordering exceeds the number of voters who rank any of the other three candidates in first or second place in their preference ordering.

Example 3.5.8.2 demonstrates the vulnerability of Bucklin's procedure to the Reinforcement paradox.

3.5.8.2 Example

This example is due to Tideman (2006, p. 205, Example 13.19). Suppose the are two districts, I and II. In District I there are 15 voters whose preference ordering among three candidates, a, b, c, are as follows:

No. of voters	Preference ordering
6	$a \succ c \succ b$
5	$b \succ a \succ c$
4	$c \succ b \succ a$

and in district II there are nine voters whose preference orderings among the same three candidates are as follows:

No. of Voters	Preference Ordering
5	$a \succ b \succ c$
4	$c \succ b \succ a$

Given these data a will be elected under Bucklin's procedure in district I (in the second counting round with 11 votes), as well as in district II (in the first counting round with five votes).

However if, *ceteris paribus*, the two districts are amalgamated into a single district, we obtain a district of 24 voters with the following preference orderings among the three candidates:

No. of voters	Preference ordering
5	$a \succ b \succ c$
6	$a \succ c \succ b$
5	$b \succ a \succ c$
8	$c \succ b \succ a$

In this amalgamated district candidate b will be elected under Bucklin's procedure (in the second counting round with 18 votes) – in violation of the reinforcement axiom.

Example 3.5.8.3 demonstrates the susceptibility of Bucklin's procedure to the no-show, twin, and truncation paradoxes.

3.5.8.3 Example

Suppose there are 101 voters whose preference orderings among four candidates, a, b, c, and d, are as follows:

No. of voters	Preference ordering
43	$a \succ b \succ c \succ d$
26	$b \succ c \succ d \succ a$
15	$c \succ d \succ b \succ a$
17	$d \succ a \succ b \succ c$

If one of the four candidates must be elected under Bucklin's procedure then candidate b would be elected because the number of voters (69) who rank b first or second in their preference ordering exceeds the number of voters who rank any of the other candidates in first or second place in their preference ordering.

Now suppose that *ceteris paribus*, 16 of the 17 voters whose top preference is d decide not to participate in the election. As a result candidate a would be elected because an absolute majority of the voters (43) rank a as their top preference. This result is preferable for all the voters whose top preference is d who thus obtain their second preference (instead of their third preference) – thereby demonstrating simultaneously both the No-Show and twin paradoxes.

To demonstrate the vulnerability of Bucklin's procedure to the truncation paradox suppose that in the above example the 43 voters whose top preference is a decide to list only their top preference. In this case a would be listed first or second by 60 voters – which is more than any other voter is listed first or second – thereby elected under Bucklin's procedure, an outcome which these 43 voters prefer to the election of b.

Example 3.5.8.3 above can also be used to demonstrate the vulnerability of Bucklin's procedure to SCC. We just saw that in this example candidate b is elected (with 69 votes in the second counting round) under Bucklin's procedure when all four candidates and 101 voters participate in the election. However, *ceteris paribus*,

a is elected under Bucklin's procedure (in the first counting round with 60 votes) if candidate d drops out of the race – thereby demonstrating the violation of the SCC postulate.

Example 3.5.8.4 demonstrates the vulnerability of Bucklin's procedure to the preference inversion paradox.

3.5.8.4 Example

Suppose that there are four voters whose preference orderings among five candidates, a, b, c, d, and e, are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c \succ d \succ e$
1	$e \succ d \succ c \succ b \succ a$
1	$b \succ e \succ c \succ a \succ d$
1	$d \succ a \succ c \succ e \succ b$

If one of the four candidates must be elected under Bucklin's procedure then candidate c would be elected because the number of voters (3) who rank c first, second, or third in their preference ordering exceeds the number of voters who rank any of the other candidates in first, second or third place in their preference ordering.

Now suppose that, *ceteris paribus*, all voters invert their preference orderings among the four candidates. In this case c, who is placed in the middle of all candidates' preference orderings, would still be elected – thus demonstrating the vulnerability of Bucklin's procedure to the preference inversion paradox.

3.5.9 Demonstrating the Paradoxes Afflicting the range Voting (RV) Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, the Range Voting (RV) procedure is vulnerable to the Condorcet winner paradox, the Condorcet loser paradox, the absolute winner paradox, the absolute loser paradox, and to the Truncation paradox.

In contrast to all other voting procedures except majority judgment (MJ) where a necessary condition to demonstrate the paradoxes afflicting them is that there exist at least three candidates, it is possible to demonstrate most of the paradoxes afflicting the RV (and MJ) procedure when there are just two candidates. The paradoxes afflicting the MJ procedure will be demonstrated in the next subsection.

Example 3.5.9.1 demonstrates simultaneously the vulnerability of the RV procedure to the first four paradoxes listed above.

3.5.9.1 Example

Suppose there are five voters, V_1, V_2, V_3, V_4, and V_5, who award the following (cardinal) grades (on a scale of 1–10) to two candidates, x and y:

Candidates /voters	V_1	V_2	V_3	V_4	V_5	Mean grade
x	2	2	2	3	10	3.8
y	1	1	1	10	7	4.0

As the mean grade of candidate y is higher than that of candidate x, candidate y is elected by the RV procedure. However, note that an absolute majority of the voters (V_1, V_2, V_3, V_5) awarded candidate x a higher grade than they awarded to candidate y, and an absolute majority of the voters (V_1, V_2, and V_3) awarded y the lowest grade. Hence candidate x is not only a Condorcet winner but also an absolute winner, whereas candidate y is not only a Condorcet loser but also an absolute loser.

Example 3.5.9.2: demonstrates the vulnerability of the RV procedure to the truncation paradox.

3.5.9.2 Example

Suppose there are seven voters, V_1–V_7, who award the following (cardinal) grades (on a scale of 1–10) to two candidates, x and y:

Candidates/voters	V_1	V_2	V_3	V_4	V_5	V_6	V_7	Mean grade
x	1	1	1	10	5	4	7	4.143
y	2	2	2	3	8	5	8	4.286

As the mean grade of candidate y is higher than that of candidate x, candidate y is elected by the RV procedure. However, as voter V_4 grades candidate x higher than y he is not satisfied with this result and will be better off if he does not grade candidate y at all, thereby demonstrating the truncation paradox. (*Ceteris paribus* if voter V_4 does not grade candidate y then this candidate will be deemed to have been awarded the lowest grade (1) by voter V_4 and, as a result, the average grade of candidate y will drop to 4.0 thus electing candidate x.)

3.5.10 Demonstrating Paradoxes Afflicting the Majority Judgment (Mj) Procedure

The paradoxes afflicting the majority judgment (MJ) procedure are discussed at length in Felsenthal and Machover (2008).

Except for being vulnerable to cyclical majorities and to strategic voting, the MJ procedure is vulnerable to the Condorcet winner paradox, the Condorcet loser paradox, the Absolute Winner paradox, the absolute loser paradox, the Truncation paradox, the Reinforcement paradox, the no-show paradox, and to the Twin paradox.

Example 3.5.10.1 demonstrates the vulnerability of the MJ procedure to the absolute winner, Condorcet winner, the absolute loser and Condorcet loser paradoxes.

3.5.10.1 Example

This example is due to Felsenthal and Machover (2008, p. 330). Suppose there are three voters, V_1, V_2, and V_3, who award the following (ordinal) grades (on a scale of A-H) to two candidates, x and y:

Candidate/voter	V_1	V_2	V_3	Median grade
x	B	C	H	C
y	A	F	G	F

As the median grade of candidate y is higher than that of candidate x, candidate y is elected by the MJ procedure. However, note that an absolute majority of the voters (V_1 and V_3) awarded candidate y a lower grade than they awarded candidate x – hence candidate x is not only a Condorcet winner but also an absolute winner, whereas candidate y is not only a Condorcet loser but also an absolute loser.

Example 3.5.10.2 demonstrates the vulnerability of the MJ procedure to the reinforcement paradox.

3.5.10.2 Example

This example is due to Felsenthal and Machover (2008, p. 327).

Suppose there are three regions, I, II, and III, in each of which 101 voters grade each of two candidates, x and y, on an ordinal scale A-D. The following lists show the distributions of grades. The figure next to a grade is the number of voters awarding that grade.

	Region I			
x:	21A	31B	48C	1D
y:	40A	11B	48C	2D
	Region II			
x:	1A	46B	14C	40D
y:	1A	45B	33C	22D
	Region III			
x:	40B	20C	41D	
y:	48B	3C	50D	

In all three elections the two candidates have equal median grades (median grade B in region I and median grade C in regions II, III), so the tie-breaking algorithm proposed by Balinski and Laraki (2007, 2011) must be used. The number of iterations required for breaking the tie in each of the three regions are 2, 7, and 2, respectively, whereupon y wins in each of the three regions.[15]

However, if the three regions are amalgamated into a single region we obtain the following distribution of grades awarded to candidates x and y by the 303 voters:

Amalgamated region:				
x:	22A	117B	82C	82D
y:	41A	104B	84C	74D

Here again candidates x and y obtain the same median grade (C), but when one breaks this tie (after 13 iterations) x wins – in violation of the Reinforcement postulate.

Example 3.5.10.3 demonstrates the vulnerability of the MJ procedure to the no-show and twin paradoxes.

3.5.10.3 Example

This example is due to Felsenthal and Machover (2008, p. 329).

Suppose that seven voters, V_1–V_7, grade two candidates, x and y, on an ordinal scale ranging between A and F, as follows:

Candidate/voter	V_1	V_2	V_3	V_4	V_5	V_6	V_7	Median grade
x	A	A	A	D	E	E	F	D
y	B	B	B	C	F	F	F	C

Here x wins. But now suppose that voters V_1 and V_2, both of whom awarded the same grades as voter V_3, and who prefer candidate y, abstain from voting. Then we get:

Candidate/voter	V_3	V_4	V_5	V_6	V_7	Median grade
x	A	D	E	E	F	E
y	B	C	F	F	F	F

Here y wins. Thus by abstaining voters V_1 and V_2 cause their favorite candidate to win – thereby demonstrating the vulnerability of the MJ procedure to the no-show

[15] To break a tie between two leading candidates who have the same median grade, one performs one or more iterations in each of which the equal median grade of the two candidates is dropped. This process continues until one reaches a situation where the candidates' median grades are no longer the same. If no such situation is reached then the tie is broken randomly. With an even number of grades Balinski and Laraki take the median to be the lower of the two middle grades.

paradox. Similarly, since V_3 prefers candidate y to x, one could expect that if, *ceteris paribus*, the two "twins" (V_1 and V_2) – who grade the two candidates in the same way as V_3 – would join the electorate, then y would certainly be elected. However, as can be seen from the first table, in this case x would be elected, thereby demonstrating the vulnerability of the MJ procedure to the twin paradox.

Example 3.5.10.4 demonstrates the vulnerability of the MJ procedure to the truncation paradox.

3.5.10.4 Example

Suppose there are seven voters, V_1–V_7, who award the following (ordinal) grades (on a scale of A- J) to two candidates, x and y:

Candidate/voter	V_1	V_2	V_3	V_4	V_5	V_6	V_7	Median grade
x	A	A	A	J	E	D	G	D
y	B	B	B	C	H	E	H	C

Here x is elected because his median grade is higher than that of y. Voter V_6 does not like this result so if, *ceteris paribus*, he decides to grade only candidate y, then candidate x would be deemed to have been awarded the lowest grade (A) by V_6 and, consequently, candidate x's median grade would drop from D to A – causing candidate y to be elected. Voter V_6 of course prefers this result – thereby demonstrating the truncation paradox.

3.5.11 Demonstrating Paradoxes Afflicting the Minimax Procedure (aka Simpson–Kramer or Condorcet's Procedure)

Except for being vulnerable to cyclical majorities and to strategic voting, the Minimax procedure (aka Condorcet's procedure or Simpson–Kramer rule) is vulnerable to the Condorcet loser, absolute loser, no-show, twin, truncation, reinforcement, preference inversion, and SCC paradoxes.

When the social preference ordering contains a top cycle it is possible that the minimax procedure will elect a Condorcet loser which may also be an absolute loser. Example 3.5.11.1 demonstrates this.

3.5.11.1 Example

Suppose there are 11 voters whose preference orderings among four candidates, a, b, c, d, are as follows:

No. of voters	Preference ordering
2	$d \succ a \succ c \succ b$
3	$d \succ b \succ a \succ c$
3	$c \succ b \succ a \succ d$
1	$b \succ a \succ c \succ d$
2	$a \succ c \succ b \succ d$

This preference profile can be depicted as the following *matrix of paired comparisons*. In such a matrix, the entry in row i and column j is the number of voters who rank candidate i ahead of candidate j.

	a	b	c	d
a	–	4	8	6
b	7	–	4	6
c	3	7	–	6
d	5	5	5	–

As an example of how the numbers in this paired comparisons matrix are calculated, the number 4 in the second column of the first row derives from the two voters in the first row of the example plus the two voters in the last row of the example who rank a ahead of b. Similarly, the number 7 in the first column of the second row derives from the three voters in the second row of the example, plus the three voters in the third row of the example, plus the one voter in the fourth row of the example, who rank b ahead of a.

As can be seen from the paired comparisons matrix, the social preference ordering in Example 3.5.11.1 contains a top cycle $[b \succ a \succ c \succ b] \succ d$, i.e., d is the Condorcet loser which happens to be also an absolute loser. However, the minimax procedure will elect d because d's worst loss margin (6) is smaller than the worst loss margin of each of the other three candidates (7, 7, 8 for a, b, c, respectively).

This example can also be used to demonstrate the vulnerability of the minimax procedure to the preference inversion paradox. If all voters invert their preference orderings then d becomes an Absolute Winner and hence is elected under the minimax procedure.

Example 3.5.11.2 demonstrates the vulnerability of the minimax procedure to the no-show and twin paradoxes.

3.5.11.2 Example

This example is due to Hannu Nurmi (private communication 22.2.2010; this example appears also in section 10.5.5 in this volume). Suppose there are 19 voters who must elect one out of four candidates, a, b, c, d and whose preference orderings among these candidates are as follows:

No. of voters	Preference ordering
5	$d \succ b \succ c \succ a$
4	$b \succ c \succ a \succ d$
3	$a \succ d \succ c \succ b$
3	$a \succ d \succ b \succ c$
4	$c \succ a \succ b \succ d$

These preference orderings can be depicted as the following paired comparisons matrix:

	a	b	c	d
a	–	10	6	14
b	9	–	12	8
c	13	7	–	8
d	5	11	11	–

Here the social preference ordering is cyclical ($c \succ a \succ d \succ b \succ c$). So according to the Minimax procedure one should elect that candidate whose worst loss is smallest. From the paired comparisons matrix it is seen that the worst loss of candidates a, b, c, d, is 13, 11, 12, and 14, respectively, so candidate b is elected.

Now suppose that, *ceteris paribus*, three of the four voters with preference ordering $c \succ a \succ b \succ d$ decide not to participate in the election. In this case the paired comparisons matrix changes as follows:

	a	b	c	d
a	–	7	6	11
b	9	–	12	5
c	10	4	–	5
d	5	11	11	–

The social preference ordering is still cyclical but the worst losses of the four candidates are now 10, 11, 12, 11 for candidates a, b, c, d, respectively, so according to the minimax procedure candidate a is elected – which is preferable from the point of view of the absent voters – thereby demonstrating the vulnerability of the minimax procedure to the no-show paradox.

We also have here an instance of the twin paradox. We have just seen that if, *ceteris paribus*, only one of the four voters whose preference orderings are $c \succ a \succ b \succ d$ participates in the election then according to the minimax procedure candidate a is elected. But if this voter's three twin brothers join the electorate then, as we have seen at the beginning of Example 3.5.11.2, candidate b is elected according to the minimax procedure – thereby demonstrating this procedure's vulnerability to the twin paradox.

Example 3.5.11.3 demonstrates the vulnerability of the minimax procedure to the truncation paradox.

3.5.11.3 Example

This example is due to Hannu Nurmi (private communication 23.2.2010; this example appears also in section 10.5.5 in this volume). As we have seen in the first part of Example 3.5.11.2, candidate b would be elected under the minimax procedure. Now suppose that, *ceteris paribus*, the four voters whose preference ordering is $c \succ a \succ b \succ d$ would decide to state only their top two preferences, c and a. This would lead to the assumption that the probability that these voters prefer b to d is equal to the probability that they prefer d to b, which would result, in turn, in the following paired comparisons matrix:

	a	b	c	d
a	–	10	6	14
b	9	–	12	6
c	13	7	–	8
d	5	13	11	–

From this paired comparisons matrix it is easy to see that candidate c's largest loss (12 against candidates b) is smallest, hence this candidate will be elected under the minimax procedure – which is certainly preferable for the voters whose top preference is c – thus demonstrating the vulnerability of the minimax procedure to the truncation paradox.

Example 3.5.11.4 demonstrates the vulnerability of the minimax procedure to the reinforcement paradox.

3.5.11.4 Example

Suppose there are two districts, one with 11 voters whose preference orderings among four candidates are as in Example 3.5.11.1 and a second district with three voters two of whom have preference ordering $d \succ a \succ b \succ c$ and the third voter has preference ordering $b \succ a \succ c \succ d$.

As we have seen in Example 3.5.11.1, candidate d will be elected in the first district, and as candidate d is the absolute winner in the second district s/he will also be elected in the second district under the minimax procedure.

Now suppose that, *ceteris paribus*, these two districts are amalgamated into one district of 14 voters having the following paired comparisons matrix:

	a	b	c	d
a	–	6	11	7
b	8	–	7	7
c	4	7	–	7
d	7	7	7	–

From this paired comparisons matrix it is easy to see that there is a tie between candidates b and d because the largest loss of both of them is smallest (7), thus according to the minimax procedure a lottery should be conducted between them – thereby demonstrating the vulnerability of the minimax procedure to the reinforcement paradox.

Example 3.5.11.5 demonstrates the vulnerability of the minimax procedure to SCC.

3.5.11.5 Example

This example is due to Fishburn (1974, p. 540). Suppose there are seven voters who are divided into three groups who have to select under the Minimax procedure one out of four candidates, a, b, c, or d, and whose preference orderings among these candidates are as follows:

Group	No. of voters	Preference ordering
G1	3	$d \succ c \succ b \succ a$
G2	2	$a \succ d \succ c \succ b$
G3	2	$b \succ a \succ d \succ c$

From this preference list we see that the social preference ordering is cyclical ($a \succ d \succ c \succ b \succ a$). It can be depicted as a (cyclical) paired comparisons matrix as follows:

	a	b	c	d
a	–	2	4	4
b	5	–	2	2
c	3	5	–	0
d	3	5	7	–

From this matrix we can see that the worst loss of candidate a is 5 (against candidate b), the worst loss of candidate b is also 5 (against candidates c, d), the worst loss of candidate c is 7 (against candidate d) and the worst loss of candidate d is 4 (against candidate a). As candidate d's loss is the smallest, this candidate would be elected under the minimax procedure.

Now suppose that, *ceteris paribus*, candidate b drops out of the race. In this case candidate a becomes the absolute winner and will be elected under the minimax procedure – in violation of SCC.

3.5.12 Demonstrating Paradoxes Afflicting Dodgson's Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, Dodgson's procedure is vulnerable to the Condorcet loser, lack of monotonicity, reinforcement, no-show, twin, truncation, preference inversion, and SCC paradoxes.

Example 3.5.12.1 demonstrates the vulnerability of Dodgson's procedure to the Condorcet loser paradox.

3.5.12.1 Example

This example is due to Fishburn (1977, p. 477). Suppose there are seven voters whose preference orderings among eight candidates, a, b, c, d, e, f, g, x, are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c \succ d \succ x \succ e \succ f \succ g$
1	$g \succ a \succ b \succ c \succ x \succ d \succ e \succ f$
1	$f \succ g \succ a \succ b \succ x \succ c \succ d \succ e$
1	$e \succ f \succ g \succ a \succ x \succ b \succ c \succ d$
1	$d \succ e \succ f \succ g \succ x \succ a \succ b \succ c$
1	$c \succ d \succ e \succ f \succ x \succ g \succ a \succ b$
1	$b \succ c \succ d \succ e \succ x \succ f \succ g \succ a$

The social preference ordering contains a top cycle $[a \succ b \succ c \succ d \succ e \succ f \succ g \succ a] \succ x$. It can be presented by the following paired comparisons matrix:

	a	b	c	d	e	f	g	x
a	–	6	5	4	3	2	1	4
b	1	–	6	5	4	3	2	4
c	2	1	–	6	5	4	3	4
d	3	2	1	–	6	5	4	4
e	4	3	2	1	–	6	5	4
f	5	4	3	2	1	–	6	4
g	6	5	4	3	2	1	–	4
x	3	3	3	3	3	3	3	–

As can easily be seen from this matrix, candidate x is a Condorcet loser as this candidate is beaten in pairwise comparisons by each of the other seven candidates. Nevertheless, candidate x will be elected in this case by Dodgson's procedure because for x to become a Condorcet winner only four preference inversions are needed (e.g., it is sufficient for any of the voters to move candidate x from fifth to first place in his preference ordering), whereas for any of the other candidates to become a Condorcet winner at least six preference inversions are needed.

This example can also be used to demonstrate the vulnerability of Dodgson's procedure to the Preference Inversion paradox. If all voters invert their preference orderings in this example then x becomes a Condorcet winner and hence is elected under Dodgson's procedure.

Example 3.5.12.2 demonstrates the vulnerability of Dodgson's procedure to lack of monotonicity. At least four candidates must exist for this to occur (Fishburn, 1982, p. 132).

3.5.12.2 Example

This example was adapted by Hannu Nurmi (private communication 15.2.2010) from Fishburn (1977, p. 478). Suppose there are 100 voters who are divided into four groups, who must elect one out of five candidates a, b, c, d, e, under Dodgson's procedure, and whose preference orderings among the candidates are as follows:

Group	No. of voters	Preference ordering
G1	42	$b \succ a \succ c \succ d \succ e$
G2	26	$a \succ e \succ c \succ b \succ d$
G3	21	$e \succ d \succ b \succ a \succ c$
G4	11	$e \succ a \succ b \succ d \succ c$

The social preference ordering has a top cycle: $[b \succ a \succ e \succ b] \succ c \succ d$. It can be depicted as the following paired comparisons matrix:

	a	b	c	d	e
a	–	37	100	79	68
b	63	–	74	79	42
c	0	26	–	68	42
d	21	21	32	–	42
e	32	58	58	58	–

For candidate a to become the Condorcet winner at least 14 voters in group G1 must change $b \succ a$ in their preference ordering to $a \succ b$, i.e., a total of 14 changes.

For candidate b to become the Condorcet winner at least nine voters from group G4 must first change $a \succ b$ to $b \succ a$ and thereafter $e \succ b$ to $b \succ e$ in their preference ordering, i.e., a total of 18 changes.

For candidate e to become the Condorcet winner at least 19 voters in group G2 must change $a \succ e$ in their preference ordering to $e \succ a$, i.e., a total of 19 changes.

Since the number of changes needed in the voters' preference orderings in order for a to become the Condorcet winner is the smallest, a would be elected under Dodgson's procedure.

Now suppose that, *ceteris paribus*, the 11 voters in group G4 *increase* their support of candidate a by changing their preference orderings from $e \succ a \succ b \succ d \succ c$ to $a \succ e \succ b \succ d \succ c$. This change can be depicted by the following paired comparisons matrix:

	a	b	c	d	e
a	–	37	100	79	79
b	63	–	74	79	42
c	0	26	–	68	42
d	21	21	32	–	42
e	21	58	58	58	–

From this matrix it is possible to see that despite the increase in a's support it would still take at least 14 persons from group G1 to change in their preference orderings $b \succ a$ to $a \succ b$ in order for a to become the Condorcet winner, whereas now for b to become the Condorcet winner only nine voters in G4 would have to change $e \succ b$ to $b \succ e$ in their preference orderings. So since the number of changes needed for b to become the Condorcet winner is smallest, b would be elected under Dodgson's procedure – thereby demonstrating lack of monotonicity.

The first part of Example 3.5.12.2 can also be used to demonstrate the vulnerability of Dodgson's procedure to the no-show and twin paradoxes. If 20 of the 21 voters in group G3 decide not to participate in the election then b becomes the Condorcet winner and will be elected according to Dodgson's procedure. The election of b is of course preferred by the members of group G3 over the election of a thus demonstrating the vulnerability of Dodgson's procedure to the no-show paradox. Adding those 20 twins back to retrieve the original profile shows that Dodgson's procedure is also vulnerable to the Twin paradox.

Example 3.5.12.2 can also be used to demonstrate the vulnerability of Dodgson's procedure to SCC. As we have seen from the paired comparisons matrix of the first part of Example 3.5.12.2 candidate a is selected by Dodgson's procedure. However if, *ceteris paribus*, candidate e drops out of the race then candidate b becomes the Condorcet winner and is elected by Dodgson's procedure – in violation of SCC.

Example 3.5.12.3 demonstrates the vulnerability of Dodgson's procedure to the Reinforcement paradox.[16]

3.5.12.3 Example

This example is due to Fishburn (1977, p. 484). Suppose there are two districts, I and II, in each of them one of four candidates, w, x, y, z, must be elected.

In district I there are seven voters, four with preference ordering $x \succ y \succ z \succ w$ and three with preference ordering $y \succ x \succ z \succ w$. Since x is here the Condorcet winner, x is elected according to Dodgson's procedure.

[16]Note that by increasing a's support, the 11 voters of group G4 obtained the election of b which for them is a less preferable alternative than the election of a. In demonstrating the non-monotonicity paradox under the other four procedures surveyed in this chapter that are susceptible to this paradox (Plurality with Runoff, Alternative Vote, Coombs, Nanson), it is exemplified not only that an original winner, w, loses after one or more voters, V_i, increase their support of w by moving w upwards in their preference ordering, but also that the voters belonging to V_i *benefit* from this because the new winner, y, is ranked higher than w in V_i's original preference ordering. However, under Dodgson's procedure it is impossible to construct such an example because when w rises in V_i's ranking, the indirect benefit, if any, goes to the candidates ranked below w in V_i's preference ordering who now find the candidates who had been ranked above w more accessible. But if V_i's initial ranking is assumed to be sincere, then it follows, by definition, that the members of V_i prefer w over any of the candidates ranked below w. So if some candidate ranked below w is elected then the members of V_i are harmed. Hence non-monotonicity under Dodgson's procedure cannot arise from considerations of strategic voting. I am grateful to Nicolaus Tideman for this insight (private communication 3.8.2011).

In district II there are 12 voters whose preference ordering are as follows:

No. of voters	Preference ordering
1	$x \succ y \succ z \succ w$
2	$y \succ x \succ z \succ w$
3	$w \succ y \succ x \succ z$
3	$z \succ w \succ y \succ x$
3	$x \succ z \succ w \succ y$

These orderings can be presented as the following paired comparisons matrix:

	w	x	y	z
w	–	6	9	3
x	6	–	4	9
y	3	8	–	6
z	9	3	6	–

Here the social preference ordering is cyclical $[w \succ y \succ x \succ z \succ w]$. For x to become the Condorcet winner only four preference inversions are needed (the two voters whose top preference is y should change their top preference to x, and one of the three voters whose top preference is w should change his top preference to x), whereas for any of the other candidates to become a Condorcet winner more than four preference inversions are needed. So according to Dodgson's procedure candidate x is elected also in district II.

Now suppose that, *ceteris paribus*, the two districts are amalgamated into a single district with 19 voters. In this case candidate y becomes the Condorcet winner and is elected according to Dodgson's procedure – thereby demonstrating its vulnerability to the Reinforcement paradox.

Example 3.5.12.4 demonstrates Dodgson's vulnerability to the Truncation paradox.

3.5.12.4 Example

Suppose there are 49 voters whose preference orderings among five candidates, a, b, c, d, e are as follows:

No. of voters	Preference ordering
11	$b \succ a \succ d \succ e \succ c$
10	$e \succ c \succ b \succ d \succ a$
10	$a \succ c \succ d \succ b \succ e$
2	$e \succ c \succ d \succ b \succ a$
2	$e \succ d \succ c \succ b \succ a$
2	$c \succ b \succ a \succ d \succ e$
1	$d \succ c \succ b \succ a \succ e$
1	$a \succ b \succ d \succ e \succ c$
10	$e \succ d \succ a \succ b \succ c$

These orderings can be presented as the following paired comparisons matrix:

	a	b	c	d	e
a	–	21	32	24	25
b	28	–	22	24	25
c	17	27	–	24	13
d	25	25	25	–	25
e	24	24	36	24	–

Here candidate d is the Condorcet winner so this candidate is elected according to Dodgson's procedure. However, if the 10 voters whose preference ordering is $e \succ d \succ a \succ b \succ c$ decide to reveal only their top preference (e) – in which case one assumes that these voters prefer candidate e over all the other four candidates and that all possible preference orderings among these candidates are equiprobable – then one obtains the following paired comparisons matrix:

	a	b	c	d	e
a	–	16	27	29	25
b	33	–	17	29	25
c	22	32	–	29	13
d	20	20	20	–	25
e	24	24	36	24	–

From this matrix we see that the social preference ordering is cyclical ($d \succ e \succ c \succ b \succ a \succ d$). So according to Dodgson's procedure candidate e is elected in this situation because for candidate e to become a Condorcet winner only three preference inversions are needed (if one of the 11 $b \succ a \succ d \succ e \succ c$ voters will change his preference ordering to $e \succ b \succ a \succ d \succ c$), whereas for any of the other candidates to become a Condorcet winner more than three preference inversions are needed – thereby demonstrating the vulnerability of Dodgson's procedure to the truncation paradox.

3.5.13 Demonstrating the Paradoxes Afflicting Black's Procedure

Since Black's procedure is a hybrid procedure (when a Condorcet winner exists it elects the Condorcet winner, and when a Condorcet winner does not exist it elects the Borda winner), it is vulnerable to the no-show, twin, truncation, reinforcement, and SCC paradoxes. Although Black's procedure is not vulnerable to the Condorcet loser paradox, it may violate Smith's (1973) Condorcet principle.

Example 3.5.13.1 demonstrates the vulnerability of Black's procedure to the no-show, twin, and truncation paradoxes.

3.5.13.1 Example

This example is partly due to Hannu Nurmi (private communication, 15.2.2010; this example appears also in section 10.5.1 in this volume). Suppose there are 16 voters whose preference orderings among five candidates, $a,\ b,\ c,\ d,\ e$, are as follows:

No. of voters	Preference ordering
3	$d \succ e \succ a \succ b \succ c$
3	$e \succ a \succ c \succ b \succ d$
4	$c \succ d \succ e \succ a \succ b$
3	$d \succ e \succ b \succ c \succ a$
3	$e \succ b \succ a \succ d \succ c$

Here d is the Condorcet winner and hence is elected under Black's procedure. Suppose now that, *ceteris paribus*, two of the voters whose preference ordering is $e \succ b \succ a \succ d \succ c$ decide not to participate in the election. As a result the social preference ordering becomes cyclical ($a \succ b = c = d \succ e \succ a$) and e emerges as the Borda winner and is therefore elected under Black's procedure. Since e is ranked first by the two absent voters, it turns out that they obtained a better outcome by not participating in the election – thereby demonstrating the vulnerability of Black's procedure to the no-show paradox.

We also have here an instance of the Twin paradox: if, *ceteris paribus*, the two absent voters decide to participate in the election and join their twin brother, then d becomes the Condorcet winner and will be elected under Black's procedure – thereby demonstrating the vulnerability of Black's procedure to the twin paradox.

Obviously, not voting at all is an extreme version of truncation and hence the above example can also be used to show that Black's procedure is vulnerable to the truncation paradox. Thus if, *ceteris paribus*, all three voters whose preference ordering is $e \succ b \succ a \succ d \succ c$ truncate their preference ordering after a, i.e., if they do not express their preferences between c and d – which would automatically be considered to mean that they prefer each of the three ranked alternatives over c and d and are indifferent between c and d – then the social preference ordering will become cyclic ($d \succ e \succ a \succ b \succ c \succ d$) and e will emerge as the Borda winner to be elected under Black's procedure – which is a preferable outcome for these voters.

The vulnerability of Borda's procedure (and hence also Black's) to the truncation paradox when a Condorcet winner does not exist initially is demonstrated in Example 3.5.5.2 above.

Example 3.5.13.2 demonstrates the vulnerability of Black's procedure to the reinforcement paradox.

3.5.13.2 Example

Suppose there are two districts, I and II. In district I there are five voters whose preference orderings among three candidates, $a,\ b$, and c, are as follows:

No. of voters	Preference ordering
2	$a \succ b \succ c$
2	$b \succ c \succ a$
1	$c \succ a \succ b$

and in District II there are nine voters whose preference orderings among these three candidates are as follows:

No. of voters	Preference ordering
5	$b \succ c \succ a$
4	$c \succ a \succ b$

The social preference ordering in district I is cyclical ($a \succ b \succ c \succ a$), so according to Borda's (and Black's) procedure candidate b, whose Borda score (6) is largest, is elected in this district. In district II candidate b is the Condorcet winner, so according to Black's procedure b is elected in this district too.

Now suppose that, *ceteris paribus*, the two districts are amalgamated into a single large district of 14 voters whose preference ordering among the three candidates are as follows:

No. of voters	Preference ordering
2	$a \succ b \succ c$
7	$b \succ c \succ a$
5	$c \succ a \succ b$

As the social preference ordering in the amalgamated district is cyclical ($c \succ a = b \succ c$) candidate c is elected in this district because his Borda score (17) is largest – thus demonstrating the vulnerability of Black's procedure to the reinforcement paradox.

The vulnerability of Black's procedure to SCC is demonstrated in Example 5.11.5 above. When all four candidates compete the social preference ordering is cyclical ($a \succ d \succ c \succ b \succ a$) so according to Black's procedure candidate d is elected because this candidate has the highest Borda score (15). But if, *ceteris paribus*, candidate b drops out of the race then candidate a becomes the Condorcet winner and is therefore elected according to Black's procedure – contrary to SCC.

Example 3.5.13.3 demonstrates the violation of Smith's (1973) Condorcet principle by Black's procedure.

3.5.13.3 Example

This example is due to Fishburn (1977, p. 480). Suppose there are five voters whose preference orderings among eight candidates a, b, c, d, e, x, y, z, are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c \succ x \succ y \succ z \succ d \succ e$
1	$e \succ a \succ b \succ x \succ y \succ z \succ c \succ d$
1	$d \succ e \succ a \succ x \succ y \succ z \succ b \succ c$
1	$c \succ d \succ e \succ x \succ y \succ z \succ a \succ b$
1	$b \succ c \succ d \succ x \succ y \succ z \succ e \succ a$

These preference orderings can be depicted as the following paired comparisons matrix:

	a	b	c	d	e	x	y	z
a	–	4	3	2	1	3	3	3
b	1	–	4	3	2	3	3	3
c	2	1	–	4	3	3	3	3
d	3	2	1	–	4	3	3	3
e	4	3	2	1	–	3	3	3
x	2	2	2	2	2	–	5	5
y	2	2	2	2	2	0	–	5
z	2	2	2	2	2	0	0	–

The social preference ordering here has a top cycle $[a \succ b \succ c \succ d \succ e \succ a] \succ x \succ y \succ z$, so according to Black's procedure one must use Borda's procedure in order to determine which of the eight candidates will be deemed the winner. The Borda counts of each of the candidates $a - e$ is 19, that of candidate x is 20, and those of candidates y and z are 15 and 10, respectively. So according to Black's procedure candidate x is elected because he has the highest Borda score. However, since Borda's procedure violates here Smith's (1973) Condorcet principle, so does Black's procedure.[17]

3.5.14 *Demonstrating Paradoxes Afflicting Copeland's Procedure*

Except for being vulnerable to cyclical majorities and to strategic voting, Copeland's procedure is vulnerable to the no-show, twin, truncation, reinforcement and SCC paradoxes.

[17] As noted above in Sect. 3.2.1.4, Smith's (1973) Condorcet principle states that if the set of candidates can be partitioned into two disjoint subsets, A and B, such that each candidate belonging to A can beat in paired comparisons each of the candidates belonging to B, then none of the candidates belonging to B ought to be elected unless all candidates in A are elected. In Example 3.5.13.3 each of candidates $a - e$ beats in paired comparisons each of the candidates x, y, z. However, Borda's procedure (and Black's) elects here candidate x although only a single candidate must be elected – in violation of Smith's Condorcet principle.

Example 3.5.14.1 demonstrates the vulnerability of Copeland's procedure to thε no-show, twin, and truncation paradoxes.

3.5.14.1 Example

Suppose there are 33 voters who must select one out of four candidates, a, b, c, oı d, and whose preference orderings among these four candidates are as follows:

No. of voters	Preference ordering
11	$a \succ b \succ c \succ d$
2	$b \succ c \succ a \succ d$
12	$b \succ c \succ d \succ a$
4	$c \succ a \succ d \succ b$
2	$d \succ a \succ b \succ c$
2	$d \succ b \succ a \succ c$

This preference list can be depicted as the following paired comparisons matrix:

	a	b	c	d
a	–	17	15	17
b	16	–	29	25
c	18	4	–	29
d	16	8	4	–

From this paired comparisons matrix we see that the social preference ordering has a top cycle $[a \succ b \succ c \succ a] \succ d$, so according to Copeland's procedure there is a tie between a, b and c.

Now suppose that, *ceteris paribus*, one of the two voters whose preference ordering is $b \succ c \succ a \succ d$ decides not to participate in the election. This change will result in the following paired comparisons matrix:

	a	b	c	d
a	–	17	15	16
b	15	–	28	24
c	17	4	–	28
d	16	8	4	–

From this matrix we can see that according to Copeland's procedure each of candidates b and c gets two points (since each of these two candidates beats two other candidates), while candidates a and d get 1.5 and 0.5 points, respectively. This result is certainly preferable from the point of view of the voter who decided not to participate, thus demonstrating the vulnerability of the Copeland's procedure to the no-show paradox.

The same example can also be used to demonstrate the vulnerability of Copeland's procedure to the twin paradox.

We have just seen that in the second part of this example one obtains a tie between candidates b and c. So one could expect, presumably, that if a twin brother of the voter with preference ordering $b \succ c \succ a \succ d$ joins the electorate (instead of abstaining), the chances of candidate b to get elected would increase. But as we have seen from the first part of this example when, *ceteris paribus*, two voters with preference ordering $b \succ c \succ a \succ d$ exist in the electorate then the chances of candidate b to get elected according to Copeland's procedure *decrease* because in this case one obtains a tie between b and two other candidates (a and c), whereas one obtains a tie between b and just one other candidate (c) when only one voter with preference ordering $b \succ c \succ a \succ d$ exists in the electorate – thus demonstrating the vulnerability of Copeland's procedure to the twin paradox.

To demonstrate the truncation paradox suppose that, *ceteris paribus*, in the first part of the above example the two voters with preference ordering $b \succ c \succ a \succ d$ would decide to reveal only their top preference. In this case one would have to assume that all the six possible preference orderings of these voters among candidates $a,\ c,\ d$ are equiprobable (or, equivalently, that they are indifferent among them) and, consequently, one would obtain the following paired comparisons matrix:

	a	b	c	d
a	–	17	16	16
b	16	–	29	25
c	17	4	–	28
d	17	8	5	–

From this paired comparisons matrix it is easy to see that according to Copeland's procedure there would be a tie between candidates b and c (each obtaining two points) – which is a preferable result from the point of view of the two $b \succ c \succ a \succ d$ voters over a tie among candidates $a,\ b,\ c$ which was obtained, *ceteris paribus*, when these voters revealed their entire preference ordering among all four candidates.

Example 3.5.14.2 demonstrates the vulnerability of Copeland's procedure to the reinforcement paradox.

3.5.14.2 Example

Suppose there are two districts, I and II. In district I there are three voters whose preference orderings among four candidates, $a,\ b,\ c$, and d, are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c \succ d$
1	$b \succ d \succ c \succ a$
1	$d \succ c \succ a \succ b$

and in district II there are two voters, one with preference ordering $b \succ d \succ c \succ a$, and the other with preference ordering $d \succ b \succ c \succ a$.

According to Copeland's procedure there is a tie between candidates b and d in each of the two districts.

However, *ceteris paribus*, if the two districts are amalgamated into a single district of five voters then one obtains the following preference list:

No. of voters	Preference ordering
1	$a \succ b \succ c \succ d$
2	$b \succ d \succ c \succ a$
1	$d \succ b \succ c \succ a$
1	$d \succ c \succ a \succ b$

This preference list can be depicted as the following paired comparisons matrix:

	a	b	c	d
a	–	2	1	1
b	3	–	4	3
c	4	1	–	1
d	4	2	4	–

From this paired comparisons matrix it is clear that candidate b is the Condorcet winner and hence is elected according to Copeland's procedure – contrary to the reinforcement axiom.

Example 3.5.11.5 can be used to demonstrate the vulnerability of Copeland's procedure to the SCC paradox. According to that example there is a tie according to Copeland's procedure between candidates a and d. However if, *ceteris paribus*, candidate b is eliminated then candidate a becomes the Condorcet winner and is elected by Copeland's procedure – in violation of the SCC postulate.

3.5.15 Demonstrating the Paradoxes Afflicting Kemeny's Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, Kemeny's procedure is vulnerable to the reinforcement, no-show, twin, truncation, and SCC paradoxes.

Example 3.5.15.1 demonstrates the vulnerability of Kemeny's procedure to the reinforcement paradox. It can also be used to demonstrate the vulnerability of Dodgson's procedure to this paradox.

3.5.15.1 Example

This example is due to Fishburn (1977, p. 484). Suppose there are two districts, I and II.

In district I there are two voters whose preference orderings among nine candidates are as follows: $x \succ y \succ a \succ b \succ c \succ d \succ e \succ f \succ g$. Here x is the Condorcet winner and hence will be elected according to Kemeny's procedure.

In district II there are seven voters whose preference orderings among the nine candidates are as follows:

No. of voters	Preference ordering
1	$y \succ x \succ a \succ b \succ c \succ d \succ e \succ f \succ g$
1	$y \succ x \succ g \succ a \succ b \succ c \succ d \succ e \succ f$
1	$y \succ x \succ f \succ g \succ a \succ b \succ c \succ d \succ e$
1	$e \succ f \succ g \succ a \succ b \succ c \succ d \succ y \succ x$
1	$d \succ e \succ f \succ g \succ a \succ b \succ c \succ y \succ x$
1	$c \succ d \succ e \succ f \succ g \succ a \succ b \succ y \succ x$
1	$x \succ b \succ c \succ d \succ e \succ f \succ g \succ a \succ y$

These preference orderings can be depicted as the following paired comparisons matrix:

	a	b	c	d	e	f	g	x	y
a	–	6	5	4	3	2	1	3	4
b	1	–	6	5	4	3	2	3	4
c	2	1	–	6	5	4	3	3	4
d	3	2	1	–	6	5	4	3	4
e	4	3	2	1	–	6	5	3	4
f	5	4	3	2	1	–	6	3	4
g	6	5	4	3	2	1	–	3	4
x	4	4	4	4	4	4	4	–	1
y	3	3	3	3	3	3	3	6	–

The social preference ordering here is cyclical: x beats each of the seven candidates $a - g$, whereas y beats x but is beaten by each of the seven candidates $a - g$. So it is clear that according to Kemeny's procedure the closest (non-cyclical) social preference ordering here is one in which x is the top-ranked candidate. (Note that x here has also the largest Borda score). So in district II too x is elected according to Kemeny's procedure.

However, in the amalgamated district (consisting of districts I and II), we obtain the following paired comparisons matrix:

	a	b	c	d	e	f	g	x	y
a	–	8	7	6	5	4	3	3	4
b	1	–	8	7	6	5	4	3	4
c	2	1	–	8	7	6	5	3	4
d	3	2	1	–	8	7	6	3	4
e	4	3	2	1	–	8	7	3	4
f	5	4	3	2	1	–	8	3	4
g	6	5	4	3	2	1	–	3	4
x	6	6	6	6	6	6	6	–	3
y	5	5	5	5	5	5	5	6	–

According to this matrix y is the Condorcet winner and hence elected under Kemeny's procedure – thereby demonstrating its vulnerability to the reinforcement paradox.

Example 3.5.15.2 demonstrates the vulnerability of Kemeny's procedure to the No-Show, Twin, and Truncation paradoxes.

3.5.15.2 Example

This example is due to Hannu Nurmi (private communication 27.2.2010 and 17.7.2011; this example appears also in section 10.5.7 in this volume). Suppose there are 19 voters whose preference orderings among four candidates, a, b, c, d, are as follows:

No. of voters	Preference ordering
5	$d \succ b \succ c \succ a$
4	$d \succ a \succ b \succ c$
4	$b \succ c \succ a \succ d$
3	$a \succ d \succ c \succ b$
3	$a \succ d \succ b \succ c$

Here a is the Condorcet winner and is therefore elected under Kemeny's procedure.

Now suppose that, *ceteris paribus*, the four $d \succ a \succ b \succ c$ voters decide not to participate in the election. As a result we obtain that the social preference ordering is cyclical [$d \succ b \succ c \succ a \succ d$], so according to Kemeny's procedure the most likely (transitive) social preference ordering is $d \succ b \succ c \succ a$ because the sum (57) associated with the pairwise comparisons of this social preference ordering is highest. So according to Kemeny's procedure d will now be elected – which the four absentee $d \succ a \succ b \succ c$ voters certainly prefer to the election of a, thereby demonstrating the vulnerability of Kemeny's procedure to the No-Show paradox.

We also have here an instance of the twin paradox. To show Kemeny's procedure vulnerability to the Twin paradox start with the 16-voter profile:

No. of voters	Preference ordering
5	$d \succ b \succ c \succ a$
1	$d \succ a \succ b \succ c$
4	$b \succ c \succ a \succ d$
3	$a \succ d \succ c \succ b$
3	$a \succ d \succ b \succ c$

Here the social preference ordering is cyclical $[d \succ b \succ c \succ a \succ d]$ and according to Kemeny's procedure the two most likely (transitive) social preference orderings are $d \succ b \succ c \succ a$ and $a \succ d \succ b \succ c$ because the sum (61) associated with the pairwise comparisons of these social preference orderings is highest. So according to Kemeny's procedure there is a tie between a and d (to be broken randomly).

Now suppose that, *ceteris paribus*, one twin brother of the $d \succ a \succ b \succ c$ voter joins the electorate, thereby, presumably, strengthening the position of d to be elected under Kemeny's procedure. But if this twin joins the electorate then a will be elected under Kemeny's procedure – thus demonstrating its vulnerability to the twin paradox. (*Ceteris paribus*, if one twin brother of the $d \succ a \succ b \succ c$ voter join the electorate then the social preference ordering will still be cyclical but according to Kemeny's procedure the most likely transitive social preference ordering will be topped by a, not by d, thereby demonstrating the vulnerability of Kemeny's procedure to the Twin paradox. We also have here an instance of the Truncation paradox. To show Kemeny's procedure vulnerability to this paradox suppose that the four voters with preference ordering $d \succ a \succ b \succ c$ list only their top preference (d). In this case one assumes that these voters are indifferent among a, b, and c, and as a result the social preference ordering becomes cyclical ($d \succ b \succ c \succ a \succ d$) and the most likely transitive social preference ordering will be topped by d, not by a, thereby demonstrating the vulnerability of Kemeny's procedure to the Truncation paradox).

Example 3.5.11.5 demonstrates the vulnerability of Kemeny's procedure to the SCC paradox. In that example candidate d is elected according to Kemeny's procedure (because the "most likely" social preference ordering according to this procedure is $d \succ c \succ b \succ a$) but if, *ceteris paribus*, candidate b is eliminated then candidate a becomes the Condorcet winner and is elected according to Kemeny's procedure – in violation of the SCC postulate.

3.5.16 Demonstrating Paradoxes Afflicting Nanson's Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, Nanson's procedure may display non-monotonicity, as well as being vulnerable to the Reinforcement, no-show, twin, truncation, and SCC paradoxes.

Example 3.5.16.1 demonstrates the vulnerability of Nanson's procedure to lack of monotonicity.

3.5.16.1 Example

This example is due to Nicolaus Tideman (private communication, 3.8.2011). Suppose there are 36 voters who must elect one out of four candidates, a, b, c, or d, under Nanson's procedure and whose preference orderings among these candidates, as well as the resultant Borda scores of the four candidates, are as follows:

No. of voters	Preference ordering	No. of voters	Preference ordering
1	$a \succ b \succ c \succ d$	2	$c \succ a \succ b \succ d$
1	$a \succ b \succ d \succ c$	1	$c \succ a \succ d \succ b$
2	$a \succ c \succ b \succ d$	3	$c \succ b \succ a \succ d$
2	$a \succ c \succ d \succ b$	2	$c \succ b \succ d \succ a$
1	$a \succ d \succ b \succ c$	1	$c \succ d \succ a \succ b$
2	$a \succ d \succ c \succ b$	2	$c \succ d \succ b \succ a$
2	$b \succ a \succ c \succ d$	1	$d \succ a \succ b \succ c$
2	$b \succ a \succ d \succ c$	1	$d \succ a \succ c \succ b$
1	$b \succ c \succ a \succ d$	0	$d \succ b \succ c \succ a$
1	$b \succ c \succ d \succ a$	2	$d \succ b \succ a \succ c$
2	$b \succ d \succ a \succ c$	1	$d \succ c \succ a \succ b$
1	$b \succ d \succ c \succ a$	2	$d \succ c \succ b \succ a$

The Borda scores of the candidates can be derived from the sum of the lines in the following paired comparisons matrix:

	a	b	c	d	Sum
a	–	16	19	20	55
b	20	–	15	20	55
c	17	21	–	20	58
d	16	16	16	–	48
Total					216

The sum of Borda scores of all four candidates is 218,[18] hence the average Borda score is 54 (216:4). According to Nanson's procedure one eliminates at the end of every counting round those candidates whose Borda score is equal to or smaller than the average score of all candidates participating in this round. Hence only candidate d is eliminated after the first round. So in the second counting round we have:

[18]Note that the sum of the Borda scores of all candidates can also be obtained by multiplying the number of voters (36 in this example) by the number of paired comparisons among the candidates (six in this example).

No. of voters	Preference ordering
4	$a \succ b \succ c$
7	$a \succ c \succ b$
8	$b \succ a \succ c$
3	$b \succ c \succ a$
5	$c \succ a \succ b$
9	$c \succ b \succ a$

which can be depicted as the following paired comparisons matrix cum Borda scores:

	a	b	c	Sum
a	–	16	19	35
b	20	–	15	35
c	17	21	–	38
Total				108

Here the sum of Borda scores of all three candidates is 108, hence their average Borda score is 36 (108:3). So according to Nanson's procedure one eliminates at the end of the second counting round both candidates a and b – thus candidate c becomes the ultimate winner.

Now suppose that, *ceteris paribus*, the voter whose preference ordering is $a \succ b \succ c \succ d$ – who is not happy with the prospect that candidate c may be elected – is motivated to *increase* his support of candidate c by changing his preference ordering to $a \succ c \succ b \succ d$. As a result of this change the Borda scores of candidates b and c change to 54 and 59, respectively, while the Borda scores of the remaining two candidates, as well as the sum of all Borda scores and the average Borda score, remain the same. So now both candidates b and d are eliminated after the first counting round. In the second counting round one obtains that the (revised) Borda scores of candidates a and c are 19 and 17, respectively, so candidate a becomes the ultimate winner – thus demonstrating that Nanson's procedure is susceptible to lack of monotonicity.

Example 3.5.12.3, which demonstrates the vulnerability of Dodgson's procedure to the reinforcement paradox, can also be used to demonstrate the vulnerability of Nanson's procedure to this paradox.

Example 3.5.16.2 demonstrates the vulnerability of Nanson's procedure to the truncation paradox.

3.5.16.2 Example

Suppose there are 43 voters divided into six groups whose preference orderings among four candidates a, b, c, d are as follows:

Group	No. of voters	Preference ordering
G1	9	$a \succ b \succ d \succ c$
G2	5	$a \succ c \succ b \succ d$
G3	2	$a \succ c \succ d \succ b$
G4	5	$b \succ a \succ c \succ d$
G5	9	$b \succ d \succ c \succ a$
G6	13	$c \succ b \succ a \succ d$

Suppose further that under Nanson's procedure with k candidates one assigns k points to the top-ranked candidate, $k-1$ points to the second-ranked candidate, ..., 1 point to the kth ranked candidate, and 0 points to any non-ranked candidate.

Given the above preference orderings and the above-mentioned point assignment, the number of points awarded to candidates a, b, c, and d in the first counting round, are 114, 134, 110, and 72, respectively. Since the average number of points is 107.5 candidate d is deleted and a second counting round is conducted. The number of points awarded to candidates a, b, c in this round is 80, 93, and 85, respectively. As the average number of points in this round is 86, both candidate a and c are deleted so candidate b is elected. However, if all voters belonging to groups G2 and G6 (who are not very happy with the election of candidate b) decide not to rank (i.e., truncate) candidate b, then the number of points awarded to candidates a, b, c, and d, are 109, 85, 92, and 72, respectively. As the average number of points in this case is 89.5, candidates b, d are deleted so candidate c is elected. This result is of course preferred by the voters in groups G2 and G6 to the election of candidate b, thereby demonstrating the susceptibility of Nanson's procedure to the truncation paradox.

Example 3.5.16.3 demonstrates the vulnerability of Nanson's procedure to the no-show and twin paradoxes.

3.5.16.3 Example

This example is due to Hannu Nurmi (private communications, 25.5.2001 and 15.2.2010; this example appears also in section 10.5.2 in this volume). Suppose there are 19 voters whose preference orderings among four candidates, a, b, c, d, are as follows:

No. of voters	Preference ordering
5	$a \succ b \succ d \succ c$
5	$b \succ c \succ d \succ a$
6	$c \succ a \succ d \succ b$
1	$c \succ b \succ a \succ d$
2	$c \succ b \succ d \succ a$

Here the Borda scores of candidates a, b, c, d are 28, 31, 37, 18, respectively, and the average Borda score is 28.5. Therefore candidates a and d are eliminated, whereupon candidate b is elected under Nanson's procedure. But if, *ceteris paribus*, one of the two last voters abstains then candidate c – the abstainer's most preferred candidate – is elected under Nanson's procedure, thus demonstrating the vulnerability of this procedure to the no-show paradox.

We also have here an instance of the twin paradox: we have just seen that if there is only one voter with preference ordering $c \succ b \succ d \succ a$ then, *ceteris paribus*, candidate c will be elected under Nanson's procedure. But if he is joined by a twin with the same preference ordering then b will be elected under Nanson's procedure, thus demonstrating the vulnerability of this procedure to the twin paradox.

Example 3.5.16.4 demonstrates the vulnerability of Nanson's procedure to SCC.

3.5.16.4 Example

This example is due to Fishburn (1977, p. 486). Suppose there are 86 voters who must elect one out of four candidates, a, b, c, or d, under Nanson's procedure and whose preference orderings are as follows:

No. of voters	Preference ordering
20	$d \succ a \succ b \succ c$
20	$d \succ b \succ c \succ a$
12	$c \succ b \succ d \succ a$
28	$a \succ c \succ b \succ d$
3	$b \succ c \succ a \succ d$
3	$c \succ b \succ a \succ d$

Accordingly, the number of Borda points awarded to candidates a, b, c, and d are 130, 127, 127, and 132, respectively – so candidates b, c are deleted and in the second counting round candidate d gets more Borda points (52) than candidate a (34) and hence d is elected.

Now suppose that, *ceteris paribus*, candidate a drops out of the race. In this case the number of Borda points awarded to candidates b, c and d are 89, 89, and 80, respectively, so there is a tie (to be broken randomly) between b and c – in violation of SCC.

3.5.17 *Demonstrating Paradoxes Afflicting Schwartz's Procedure*

Except for being vulnerable to cyclical majorities and to strategic voting, Schwartz's procedure is vulnerable to the reinforcement, no-show, twin, truncation, and the pareto-dominated candidate paradoxes.

Example 3.5.17.1 demonstrates the vulnerability of Schwartz's procedure to the reinforcement paradox.

3.5.17.1 Example

This example is due to Fishburn (1977, p. 483). Suppose there are two districts, I and II. In district I there are five voters, three of whom have preference ordering $x \succ y \succ w \succ z$ and the remaining two voters have preference ordering $z \succ y \succ w \succ x$. Since x constitutes here the top preference of an absolute majority of the voters, x will be elected in district I according to Schwartz's procedure.

In district II there are four voters: one with preference ordering $y \succ x \succ z \succ w$, one with preference ordering $w \succ y \succ x \succ z$, one with preference ordering $z \succ w \succ y \succ x$, and one with preference ordering $x \succ z \succ w \succ y$. The social preference ordering here is cyclical [$z \succ w \succ y \succ x \succ z$] so all four candidates should be in the choice set in district II according to Schwartz's procedure.

It would therefore be reasonable to assume that if, *ceteris paribus*, the two districts are amalgamated into a single district of nine voters, then x should be in the choice set of the amalgamated district according to Schwartz's procedure. However, in the amalgamated district y becomes the Condorcet winner and hence is the only candidate in the choice set according to Schwartz's procedure – thus demonstrating its vulnerability to the Reinforcement paradox.

Example 3.5.17.2 demonstrates the vulnerability of Schwartz's procedure to the no-show and twin paradoxes. Unlike the demonstration of these paradoxes under other procedures, in order to demonstrate the vulnerability of Schwartz's procedure to these paradoxes one must assume whether the voters are risk-neutral, risk-averse, or risk-seeking. I shall assume that the voters are risk-neutral, i.e., when only the voters' ordinal (but not cardinal) preferences are known, I assume that a voter whose ordinal preferences between three candidates, a, b, c is $a \succ b \succ c$ will be indifferent between obtaining a tie between these three candidates which will be broken randomly and the election of candidate b with certainty. Similarly, I assume that if this voter's ordinal preferences among four candidates is $b \succ c \succ d \succ a$ he would prefer the election of candidate c with certainty than to obtain a tie among all four candidates which will be broken randomly. Using different examples it is of course possible to demonstrate these paradoxes also when one assumes that the voters are risk-averse or risk-seeking.

3.5.17.2 Example

This example is due to Hannu Nurmi (private communication, 1.3.2010; this example appears also in section 10.5.4 in this volume). Suppose there are 100 voters whose preference orderings among four candidates, a, b, c, d, are as follows:

No. of voters	Preference ordering
23	$a \succ b \succ d \succ c$
28	$b \succ c \succ d \succ a$
49	$c \succ d \succ a \succ b$

Here the social preference ordering is cyclical [$a \succ b \succ c \succ d \succ a$] and according to Schwartz's procedure all four candidates belong to the choice set.

Now suppose that, *ceteris paribus*, four of the 28 $b \succ c \succ d \succ a$ voters decide not to participate in the election. In this case c becomes the Condorcet winner – which the absentee voters certainly prefer over a tie among all candidates that will be broken randomly – thereby demonstrating the vulnerability of Schwartz's procedure to the No-Show paradox.

We have here also a demonstration of the twin paradox. We just saw that, *ceteris paribus*, if there are only 24 voters with preference ordering $b \succ c \succ d \succ a$ then candidate c is the Condorcet winner and is the only candidate belonging to the choice set according to Schwartz's procedure. But if, *ceteris paribus*, one adds another four twins with preference ordering $b \succ c \succ d \succ a$ then Schwartz's choice set includes all candidates – which is a less preferable outcome for these voters, thus demonstrating the vulnerability of Schwartz's procedure to the Twin paradox.

To demonstrate the vulnerability of Schwartz's procedure to the Truncation paradox we use again Example 3.5.13.1. In the first part of this example we obtained that candidate d is the Condorcet winner and hence is the sole candidate belonging to the Schwartz set. But, *ceteris paribus*, when the two voters whose preference ordering is $e \succ b \succ a \succ d \succ c$ decide not to reveal their last two preferences (thereby assuming that the probability that they prefer d to c is equal to the probability they prefer c to d), one obtains the following expected paired comparisons matrix:

	a	b	c	d	e
a	–	6	6	4	0
b	4	–	6	4	0
c	4	4	–	5	2
d	6	6	5	–	6
e	10	10	8	4	–

As can be seen from this matrix only candidates d, e belong to the Schwartz set (because each of these candidates either beats or ties with each of the other three candidates) – which is a preferred outcome for the above-mentioned two truncating voters over the certain election of candidate d – thereby demonstrating the vulnerability of Schwartz's procedure to the Truncation paradox.

This preference matrix can also be used to demonstrate the vulnerability of Schwartz's procedure to the SCC paradox. We have just seen that according to this preference matrix only candidates d, e belong to the Schwartz set. However, if *ceteris paribus*, candidate c is eliminated (by deleting the row c and column c from this matrix) then candidate d becomes the Condorcet winner and is elected by Schwartz's procedure – in violation of the SCC postulate.

Example 3.5.17.3 demonstrates the vulnerability of Schwartz's procedure to the pareto-dominated candidate paradox.

3.5.17.3 Example

This example is due to Fishburn (1973, p. 89; 1977, p. 478). Suppose there are three voters whose preference orderings among four candidates, a, b, c, d are as follows:

No. of voters	Preference ordering
1	$a \succ b \succ c \succ d$
1	$d \succ a \succ b \succ c$
1	$c \succ d \succ a \succ b$

Here the social preference ordering is cyclical ($a \succ b \succ c \succ d \succ a$) and according to Schwartz's procedure all four candidates belong to the choice set – this despite the fact that candidate b is dominated by a (because all voters prefer a to b) – thus demonstrating the vulnerability of this procedure to the pareto-dominated candidate paradox.

3.5.18 Demonstrating Paradoxes Afflicting Young's Procedure

Except for being vulnerable to cyclical majorities and to strategic voting, Young's procedure is vulnerable to the Condorcet loser, absolute loser, reinforcement, no-show, twin, truncation, preference inversion, and SCC paradoxes.

Example 3.5.11.1 can be used to demonstrate the vulnerability of Young's procedure to electing not only a Condorcet loser but also an Absolute Loser. In that example candidate d is an absolute loser (and hence also a Condorcet loser), but under Young's procedure d will be elected because for d to become a Condorcet winner only two voters must be removed from the 11-voter electorate (any two voters whose last preference is d), whereas for each of the other three candidates more than two voters must be removed in order for them to become a Condorcet winner.

Example 3.5.11.1 can also be used to demonstrate the vulnerability of Young's procedure to the preference inversion paradox because, as we have already seen, if all voters in Example 3.5.11.1 invert their preference orderings then d becomes an Absolute Winner and hence is elected under Young's procedure.

Example 3.5.12.3 can be used, *mutatis mutandis*, to demonstrate the vulnerability of Young's procedure to the reinforcement paradox. In that example candidate x is a Condorcet winner in district I and hence is elected in this district according to Young's procedure too. To become the Condorcet winner in district II only five voters must be removed (any five voters who prefer y to x), whereas for any of the other candidates to become a Condorcet winner in district II more than five voters must be removed. So according to Young's procedure candidate x is elected also in district II. But, as was demonstrated in Example 3.5.12.3, in the amalgamated district with 19 voters candidate y becomes the Condorcet winner and is therefore elected also according to Young's procedure – thereby demonstrating its vulnerability to the reinforcement paradox.

Example 3.5.18.1 demonstrates the vulnerability of Young's procedure to the no-show, twin, truncation, and SCC paradoxes.

3.5.18.1 Example

This example is due to Hannu Nurmi (private communication 22.2.2010). Suppose there are 39 voters whose preference orderings among five candidates, a, b, c, d, e, are as follows:

No. of voters	Preference ordering
11	$b \succ a \succ d \succ e \succ c$
10	$e \succ c \succ b \succ d \succ a$
10	$a \succ c \succ d \succ b \succ e$
2	$e \succ c \succ d \succ b \succ a$
2	$e \succ d \succ c \succ b \succ a$
2	$c \succ b \succ a \succ d \succ e$
1	$d \succ c \succ b \succ a \succ e$
1	$a \succ b \succ d \succ e \succ c$

These preference orderings can be depicted as the following paired comparisons matrix:

	a	b	c	d	e
a	–	11	22	24	25
b	28	–	12	24	25
c	17	27	–	24	13
d	15	15	15	–	25
e	14	14	26	14	–

The social preference ordering here is cyclical ($c \succ b \succ a \succ d \succ e \succ c$). The minimal number of voters one must remove in order for any of the five candidates to become a Condorcet winner is 12 (the 10 voters whose top preference is a and the two voters whose top preference is c) in order for e to become the Condorcet winner. So e is elected according to Young's procedure given this profile.

Now suppose that, *ceteris paribus*, 10 new voters whose preference ordering is $e \succ d \succ a \succ b \succ c$ join the electorate – thus presumably strengthening e's position. However, in this case we obtain the following paired comparisons matrix:

	a	b	c	d	e
a	–	21	32	24	25
b	28	–	22	24	25
c	17	27	–	24	13
d	25	25	25	–	25
e	24	24	36	24	–

which shows that candidate d is the Condorcet winner, hence the 10 added voters are better off abstaining – thus demonstrating the vulnerability of Young's procedure to the No-Show paradox.[19] Obviously twins are not always welcome here.

However, if the 10 added voters reveal only their top preference (e), then we obtain the following paired comparisons matrix:

	a	b	c	d	e
a	–	16	27	29	25
b	33	–	17	29	25
c	22	32	–	29	13
d	20	20	20	–	25
e	24	24	36	24	–

Here candidate $e$will be elected according to Young's procedure because for e to become the Condorcet winner in this case only two voters must be removed (any two voters whose bottom preference is e), whereas for any of the other candidates to become a Condorcet winner more than two voters must be removed – thus demonstrating that Young's procedure is vulnerable to the truncation paradox.

To demonstrate the vulnerability of Young's procedure to SCC let us look again at the paired comparison matrix of the 39 voters at the beginning of this example. We saw that given this matrix candidate e is elected under Young's procedure. Now suppose that, *ceteris paribus*, candidate b decides to withdraw from the race. But if, as a result, we cross out row b and column b in the paired comparison matrix, we see that candidate a becomes the Condorcet winner and hence elected by Young's procedure – thereby demonstrating its vulnerability to SCC.

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[19]The added 10 voters also demonstrate that Young's procedure violates what Pérez (1995, p. 143) has called the *Monotonicity property in face of new voters*. This property requires that if candidate x is chosen in a given situation and then, *ceteris paribus*, a new voter is added whose top preference is x, then: (1) x must remain chosen for *Weak Monotonicity* to be satisfied, and (2) x must remain chosen and no one not chosen before should be chosen now in order for *Monotonicity* to be satisfied.

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