Questions for the Final Exam

The final exam is on **Wed, Dec 18, 8:00 AM - 10:00 AM in Skinner 1115**. You will have to answer questions 1 & 11. For the remaining questions, I will select 3 of the 5 point questions and 3 of the 15 point questions. You can think about the answers to these· questions for the next week, but you must write the answers during the exam period (you will not be allowed to use any notes). Good luck!

1. (20 points) A question involving using epistemic models to describe the change in knowledge in an social interactive situation.

2. In this problem we consider a possible definition of common belief, analogous to the definition of common knowledge. Suppose there are two agents and a belief model $\langle W, \{R_1, R_2\}, V\rangle$ where R_1 and R_2 are serial, transitive and Euclidean relations. Let $R_B = (R_1 \cup R_2)^+$, where R^+ is the transitive closure of R (the smallest transitive relation containing R). Define the common belief operator C^B as follows:

$$\mathcal{M}, w \models C^B\varphi \text{ iff for each } v \in W, \text{ if } wR_Bv \text{ then } \mathcal{M}, v \models \varphi$$

   (a) (5 points) Provide a **KD45** model $\mathcal{M} = \langle W, \{R_1, R_2\}, V\rangle$ and a state $w \in W$ where $\mathcal{M}, w \models B_1(C^B p)$ but $\mathcal{M}, w \models \neg C^B p$ (i.e., a state where agent 1 believes that p is commonly believed, but p is, in fact, not commonly believed).

   (b) (5 points) Provide an example that shows that negative introspection for common belief ($\neg C^B\varphi \to C^B\neg C^B\varphi$) is not valid

3. (15 points) For a Bayesian model with a common prior $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \pi\rangle$, prove that for each $i \in \mathcal{A}$, $\pi(E \mid B_i^p(E)) \geq p$.

4. (15 points) Explain why Aumann's original agreeing to disagree theorem follows from Samet's generalized agreeing to disagree theorem (see slide 21 of the lecture "Common Knowledge, Belief and Agreeing to Disagree"). *Hint: fix an event $E \subseteq W$ and for each agent i, let the decision function $\mathbf{d}_i$ be defined as follows: $\mathbf{d}_i(w) = \pi(E \mid [w]_i)$ (the posterior probability of E for agent i at state w). Prove that $\mathbf{d}$ satisfies the ISTP.*

5. (15 points) We have argued that $K_i\varphi \to K_j\varphi$ is valid on a frame $\langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ iff for each $i, j \in \mathcal{A}$, $R_j \subseteq R_i$. Find a property on frames $\langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ that guarantees that $K_i\varphi \to K_iK_j\varphi$ is valid.

6. Recall that an **epistemic-plausibility model** is a tuple

$$\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$$

where W is a non-empty set of states, for each $i \in \mathcal{A}$, $\sim_i$ is an equivalence relation on W, for each $i \in \mathcal{A}$, $\preceq_i$ is reflexive, transitive, and *well-founded* (every subset $X \subseteq W$ has a $\preceq$-minimal element), and $V : \mathsf{At} \to \wp(W)$ is a valuation function. In addition, the following two properties are satisfied:

(a) *plausibility implies possibility*: if $w \preceq_i v$ then $w \sim_i v$.

(b) *locally-connected*: if $w \sim_i v$ then either $w \preceq_i v$ or $v \preceq_i w$.

Let $\mathcal{L}_{KB}$ be the modal language defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi \mid B^\varphi\psi \mid [\preceq_i]\varphi \mid B^s\varphi$$

with $p \in \mathsf{At}$. Recall that $Min_{\preceq_i}(X) = \{v \in X \mid v \preceq_i w \text{ for all } w \in X\}$ (which is always non-empty since $\preceq_i$ is well-founded). Truth for the modal operators is defined as follows:

- $\mathcal{M}, w \models K_i\varphi$ iff for all $v \in W$, if $w \sim_i v$ then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B_i^\varphi\psi$ iff for all $v \in Min_{\preceq_i}([\![\varphi]\!]_\mathcal{M} \cap [w]_i)$, $\mathcal{M}, v \models \psi$
- $\mathcal{M}, w \models [\preceq_i]\varphi$ iff for all $v \in W$, if $v \preceq_i w$ then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B^s\varphi$ iff $[\![\varphi]\!]_\mathcal{M} \cap [w]_i \neq \emptyset$ and $[\![\varphi]\!]_\mathcal{M} \preceq_i [\![\neg\varphi]\!]_\mathcal{M}$

where $[\![\varphi]\!]_\mathcal{M} = \{w \mid \mathcal{M}, w \models \varphi\}$.

(a) Prove that the following two formulas are valid on every epistemic-plausibility model (recall that $L_i\varphi$ is defined to be $\neg K_i\neg\varphi$ and $B_i\varphi$ is $B^\top\varphi$):

- (15 points) $B_i^s\varphi \leftrightarrow B_i\varphi \wedge K_i(\varphi \to [\preceq_i]\varphi)$
- (15 points) $(B^\varphi\alpha \wedge \neg B^\varphi\neg\beta) \to B^{\varphi\wedge\beta}\alpha$ is valid.

7. (5 points) What is the Ramsey test, and what did Peter Gärdnefors prove about the Ramsey test?

8. (5 points) Let $C_G\varphi$ be "common knowledge among group G that φ is true". Is $C_{G_1\cup G_2}\varphi \to C_{G_1}\varphi \wedge C_{G_2}\varphi$ valid? Is $C_{G_1}\varphi \wedge C_{G_2}\varphi \to C_{G_1\cup G_2}\varphi$ valid?

9. (5 points) State the contraction principle (explain what the principle says) and discuss one of Hansson's counterexamples.

10. (15 points) Given a probability measure P defined on a σ-algebra Σ. Given the definition of a P-stable^r set. Prove that for all $X \in \Sigma$ with X non-empty and P-stable^r: If $P(X) < 1$, then there is no non-empty $Y \subseteq X$ with $Y \in \Sigma$ and $P(Y) = 0$.

11. (20 points) Choose one of the following paradoxes, explain the paradox and briefly explain a proposed “solution”: Surprise Examination Paradox, Fitch’s Paradox, Absent-Minded Driver Problem, Brandenburger-Keisler Paradox, or the Preface Paradox. Your explanation should not be more than 3-5 paragraphs.