Reasoning about Knowledge and Beliefs

Lecture 18

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Puzzles about *interactive* knowledge and beliefs

$K_i E$: "i knows that E"

$K_i K_j E$: "i knows that j knows that E"

Alternative history...

J. Harsanyi. *Games with incomplete information played by "Bayesian" players I-III. Management Science Theory* **14***: 159-182, 1967-68.*

Robert Aumann. *Agreeing to Disagree*. Annals of Statistics **4** (1976).

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R. Aumann. *Interactive Epistemology I & II*. International Journal of Game Theory (1999).

P. Battigalli and G. Bonanno. *Recent results on belief, knowledge and the epistemic foundations of game theory*. Research in Economics (1999).

R. Myerson. *Harsanyi's Games with Incomplete Information*. Special 50th anniversary issue of *Management Science*, 2004.

Harsanyi Type Space

John C. Harsanyi, nobel prize winner in economics, developed a theory of games with **incomplete information**.

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1. incomplete information: uncertainty about the *structure* of the game (outcomes, payoffs, strategy space)
2. imperfect information: uncertainty *within the game* about the previous moves of the players

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4. but this is a new parameter, and so on....

Harsanyi's Problem

A (game-theoretic) **type** of a player summarizes everything the player knows privately at the beginning of the game which could affect his beliefs about payoffs in the game and about all other players' types.

(Harsanyi argued that all uncertainty in a game can be equivalently modeled as uncertainty about payoff functions.)

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- imperfect information about the play of the game
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- imperfect information about the play of the game
- incomplete information about the structure of the game
- strategic information (what will the other players do?)
- higher-order information (what are the other players thinking?)

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Formally, a game is described by its strategy sets and payoff functions. But in real life, may other parameters are relevant; there is a lot more going on. Situations that substantively are vastly different may nevertheless correspond to precisely the same strategic game.... The difference lies in the attitudes of the players, in their expectations about each other, in custom, and in history, though the rules of the game do not distinguish between the two situations. *(pg. 72)*

R. Aumann and J. H. Dreze. *Rational Expectations in Games*. American Economic Review 98 (2008), pp. 72-86.

The Epistemic Program in Game Theory

"...the analysis constitutes a fleshing-out of the textbook interpretation of equilibrium as 'rationality plus correct beliefs.'...this suggests that equilibrium behavior cannot arise out of strategic reasoning alone. "

E. Dekel and M. Siniscalchi. *Epistemic Game Theory*. manuscript, 2013.

A. Brandenburger. *The Power of Paradox*. International Journal of Game Theory, 35, pgs. 465 - 492, 2007.

EP and O. Roy. *Epistemic Game Theory*. Stanford Encyclopedia of Philosophy, forthcoming, 2013.

Doesn't such talk of what Ann believes Bob believes about her, and so on, suggest that some kind of self-reference arises in games, similar to the well-known examples of self-reference in mathematical logic.

A. Brandenburger and H. J. Keisler. *An Impossibility Theorem on Beliefs in Games*. *Studia Logica* (2006).

A Paradox

Ann believes that Bob's strongest belief is
that Ann believes that Bob's strongest belief is false.

Does Ann believe that Bob's strongest belief is false?

$*$ A **strongest belief** is a belief that implies all other beliefs.

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So, the answer is no.

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So, the answer must be yes.

- strongest belief

- strongest belief
- weakest belief

- strongest belief
- weakest belief
- craziest belief

- strongest belief
- weakest belief
- craziest belief
- all of Bob's belief

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- What exactly does "all possible" mean?
  (Complete, Canonical, Universal)
- Who cares?

Who Cares?

A. Brandenburger and E. Dekel. *Hierarchies of Beliefs and Common Knowledge*. Journal of Economic Theory (1993).

A. Heifetz and D. Samet. *Knoweldge Spaces with Arbitrarily High Rank*. Games and Economic Behavior (1998).

L. Moss and I. Viglizzo. *Harsanyi type spaces and final coalgebras constructed from satisfied theories*. EN in Theoretical Computer Science (2004).

A. Friendenberg. *When do type structures contain all hierarchies of beliefs?*. working paper (2007).

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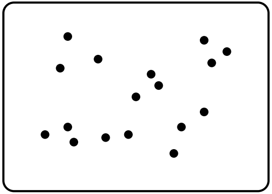
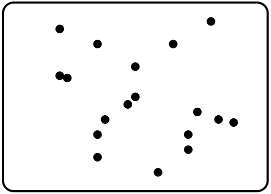
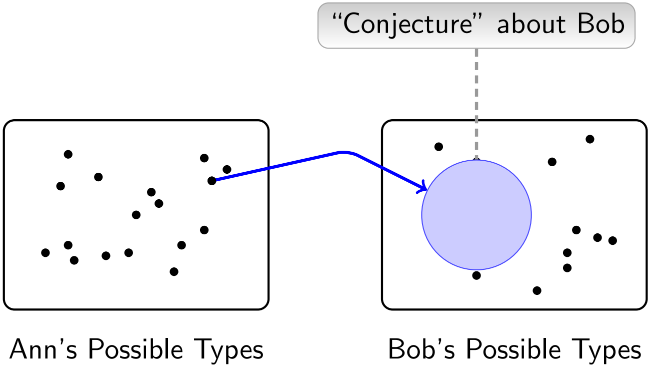
A. Brandenburger, A. Friedenberg, H. J. Keisler. *Admissibility in Games*. Econometrica (2008).



Ann's Possible Types



Bob's Possible Types


"Conjecture" about Bob
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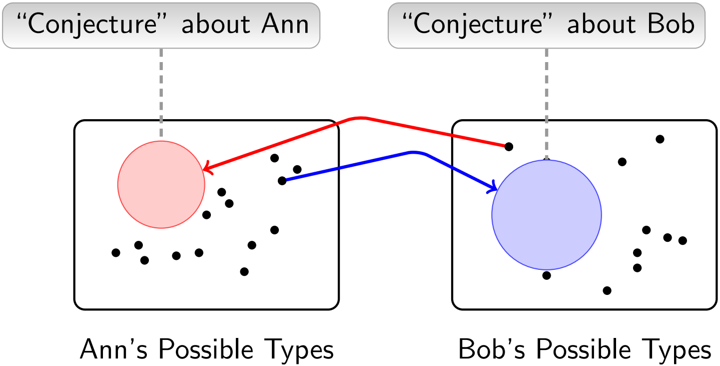
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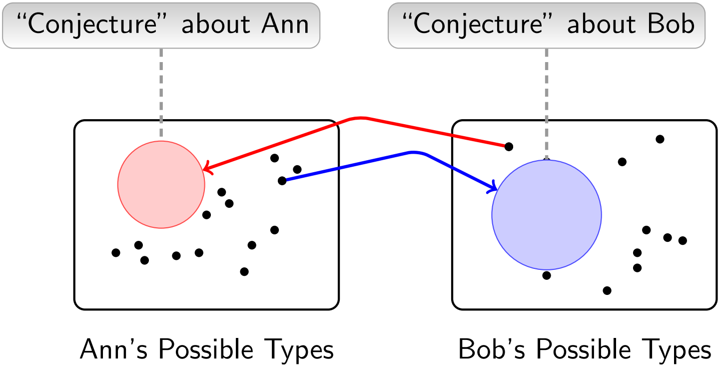

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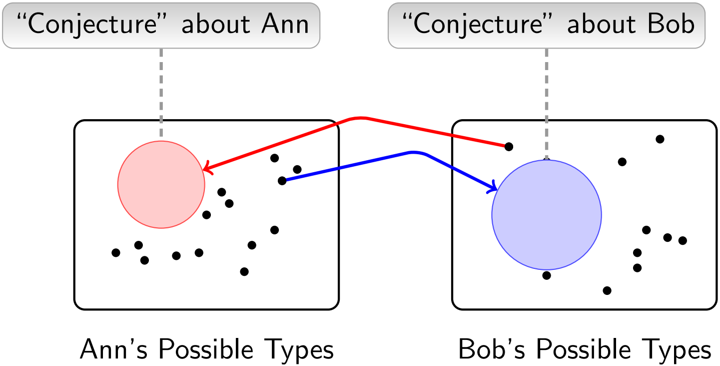

Is there a space where every *possible* conjecture is considered by *some* type? It depends...

S. Abramsky and J. Zvesper. *From Lawvere to Brandenburger-Keisler: interactive forms of diagonalization and self-reference*. Proceedings of LOFT 2010.

EP. *Understanding the Brandenburger Keisler Pardox*. Studia Logica (2007).

Impossibility Results

Language: the (formal) language used by the players to formulate conjectures about their opponents.

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Completeness: A model is **complete for a language** if every (consistent) statement in a player's language about an opponent is *considered* by some type.

Qualitative Type Spaces: $\langle T_a, T_b, \lambda_a, \lambda_b \rangle$

$\lambda_a : T_a \rightarrow \wp(T_b)$
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$\lambda_b : T_b \rightarrow \wp(T_a)$

x **believes** a set $Y \subseteq T_b$ if $\lambda_a(x) \subseteq Y$

x **assumes** a set $Y \subseteq T_b$ if $\lambda_a(x) = Y$

Impossibility Results

Impossibility 1 There is no complete interactive belief structure for the *powerset language*.

Proof. Cantor: there is no onto map from X to the nonempty subsets of X.

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Impossibility 2 (Brandenburger and Keisler) There is no complete interactive belief structure for *first-order logic*.

Suppose that $\mathcal{C}_A \subseteq \wp(T_A)$ is a set of *conjectures* about Ann and $\mathcal{C}_B \subseteq \wp(T_B)$ a set of conjectures about Bob states.

Assume For all $X \in \mathcal{C}_A$ there is a $x_0 \in T_A$ such that

1. $\lambda_A(x_0) \neq \emptyset$: "in state x_0, Ann has consistent beliefs"
2. $\lambda_A(x_0) \subseteq \{y \mid \lambda_B(y) = X\}$: "in state x_0, Ann believes that Bob's strongest belief is that X"

Lemma. Under the above assumption, for each $X \in \mathcal{C}_A$ there is an x_0 such that

$$x_0 \in X \text{ iff there is a } y \in T_B \text{ such that } y \in \lambda_A(x_0) \text{ and } x_0 \in \lambda_B(y)$$

Claim. $x_0 \in X$ iff $\exists y \in T_B$, $y \in \lambda_A(x_0)$ and $x_0 \in \lambda_B(y)$

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$\neg\varphi(x) := \forall y(R_A(x, y) \to \neg R_B(y, x))$: "Ann believes that Bob's strongest belief is *false*."

Proof of the Theorem

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	iff (Lemma)	there is a $y \in T_B$ with $y \in \lambda_A(x_0)$ and $x_0 \in \lambda_B(y)$

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2. $\lambda_A(x_0) \subseteq \{y \mid \lambda_B(y) = X\}$: At x_0, Ann believes that Bob's strongest belief is that $X = \{x \mid \neg\varphi(x)\}$ (i.e., Ann believes that Bob's strongest belief is that Ann believes that Bob's strongest belief is false.)

$\neg\varphi(x_0)$ is true	iff (def. of X)	$x_0 \in X$
	iff (Lemma)	there is a $y \in T_B$ with $y \in \lambda_A(x_0)$ and $x_0 \in \lambda_B(y)$
	iff (def. of $\varphi(x)$)	$\varphi(x_0)$ is true.