Reasoning about Knowledge and Beliefs

Lecture 17

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How do qualitative and quantitative belief relate to each other?

H. Leitgeb. *Reducing belief simpliciter to degrees of belief*. Annals of Pure and Applied Logic, 16:4, pgs. 1338 - 1380, 2013.

In view of the fact that we have a reasonably clear picture of what the logics of qualitative and quantitative belief are like, what conclusions can we draw form this on how qualitative and quantitative belief ought to relate to each other, assuming that they satisfy their respective logics? How do they relate to each other in the case of an agent who is a perfect reasoner?

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The Nihilistic proposal: "...no explication of belief is possible within the confines of the probability model."

Leitgeb's Bridge Principle

$$\text{If } Bel(A) \text{ then } P(A) > r$$

where 'r' denotes again a threshold value that is determined contextually in some way.

Quantitative Belief

Let W be a set of states and $\mathfrak{A}$ a σ-algebra: $\mathfrak{A} \subseteq \wp(W)$ such that

- $W, \emptyset \in \mathfrak{A}$
- if $X \in \mathfrak{A}$ then $W - X \in \mathfrak{A}$
- if $X, Y \in \mathfrak{A}$ then $X \cup Y \in \mathfrak{A}$
- if $X_0, X_1, \ldots \in \mathfrak{A}$ then $\bigcup_{i \in \mathbb{N}} X_i \in \mathfrak{A}$.

Quantitative Belief

$P : \mathfrak{A} \to [0, 1]$ satisfying the usual constraints

- $P(W) = 1$
- (finite additivity) If $X_1, X_2 \in \mathfrak{A}$ are pairwise disjoint, then $P(X_1 \cup X_2) = P(X_1) + P(X_2)$

$P(Y|X) = \frac{P(Y \cap X)}{P(X)}$ whenever $P(X) > 0$. So, $P(Y|W)$ is $P(Y)$.

- P is countably additive (σ-additive): if $X_1, X_2, \ldots, X_n, \ldots$ are pairwise disjoint members of $\mathfrak{A}$, then $P(\bigcup_{n \in \mathbb{N}} X_n) = \sum_{n \in \mathbb{N}} P(X_n)$

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- $Bel(W)$
- For all $X, Y \in \mathfrak{A}$, if $Bel(X)$ and $X \subseteq Y$ then $Bel(Y)$
- For all $X, Y \in \mathfrak{A}$, if $Bel(X)$ and $Bel(Y)$ then $Bel(X \cap Y)$
- For $\mathcal{Y} = \{Y \in \mathfrak{A} \mid Bel(Y)\}$, $\bigcap \mathcal{Y} \in \mathfrak{A}$ and $Bel(\bigcap \mathcal{Y})$
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"We will interpret such conditional beliefs in suppositional terms: they are beliefs that the agent has *under the supposition of certain propositions*, where the only type of supposition that we will be concerned with will be supposition *as a matter of fact*, that is, suppositions which are usually expressed in the indicative, rather than the subjunctive mood: *Suppose that X is the case. Then I believe that Y is the case.*"

Let Bel_X be the set of propositions that the agent believes conditional on X, write $Bel(Y|X)$ when $Y \in Bel_X$.

B1 (Reflexivity) If $\neg Bel(\neg X|W)$, then $Bel(X|X)$.

B2 (One Premise Logical Closure) If $\neg Bel(\neg X|W)$, then for all $Y, Z \in \mathfrak{A}$: if $Bel(Y|X)$ and $Y \subseteq Z$, then $Bel(Z|X)$

B3 (Finite Conjunction) If $\neg Bel(\neg X|W)$, then for all $Y, Z \in \mathfrak{A}$: if $Bel(Y|X)$ and $Bel(Z|X)$, then $Bel(Y \cap Z|X)$.

B4 (General Conjunction) If $\neg Bel(\neg X|W)$, then for $\mathcal{Y} = \{Y \in \mathfrak{A} \mid Bel(Y|X)\}$, $\bigcap \mathcal{Y} \in \mathfrak{A}$ and $Bel(\bigcap \mathcal{Y}|X)$

B5 (Consistency) $\neg Bel(\emptyset|X)$

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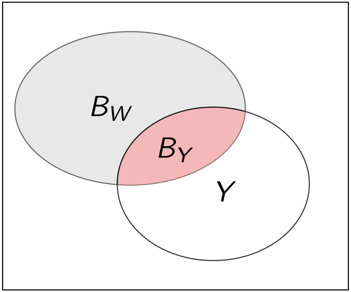
Variant of B6

For all $Y \in \mathfrak{A}$ such that $Y \cap B_W \neq \emptyset$: For all $Z \in \mathfrak{A}$, $Bel(Z|Y)$ iff $Z \supseteq Y \cap B_W$.

For all $Y \in \mathfrak{A}$ such that $Y \cap B_W \neq \emptyset$: $B_Y = Y \cap B_W$



BP1^r For all $Y \in \mathfrak{A}$ such that $Y \cap B_W \neq \emptyset$ and $P(Y) > 0$: For all $Z \in \mathfrak{A}$, if $Bel(Z|Y)$, then $P(Z|Y) > r$

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"So, it follows that $P(B_W \mid W) = P(B_W) > r$.
Therefore,...,having a subjective probability of more than r is a necessary condition for a proposition to be believed absolutely, although it will become clear later that this is not necessarily a sufficient condition."

P-stability^r

Definition. Let P be a probability measure on $\mathfrak{A}$ over W, let $0 \leq r < 1$. For all $X \in \mathfrak{A}$:

X is P-stable^r if and only if for all $Y \in \mathfrak{A}$ with $Y \cap X \neq \emptyset$ and $P(Y) > 0$: $P(X|Y) > r$.

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- If $P(X) = 1$, then X is P-stable^r.
- There are P-stable^r sets with $0 < P(X) < 1$.
- X is P-stable^r iff for all $Y, Z \in \mathfrak{A}$ such that $Y \neq \emptyset$, $Y \subseteq X$ and where $Z \subseteq \neg X$, $P(Z) > 0$ it holds that:

$$P(Y) > \frac{r}{1-r} P(Z)$$

Observation. For all $X \in \mathfrak{A}$ with X non-empty and P-stable^r: If $P(X) < 1$, then there is no non-empty $Y \subseteq X$ with $Y \in \mathfrak{A}$ and $P(Y) = 0$.

Theorem. Let *Bel* be a class of ordered pairs of members of a σ-algebra $\mathfrak{A}$, let $P : \mathfrak{A} \to [0,1]$, and let $0 \leq r < 1$. Then the following two statements are equivalent:

1. P and *Bel* satisfy $P1$, $B1 - B6$, and $BP1r$.
2. P satisfies $P1$, and there is an $X \in \mathfrak{A}$, such that X is a non-empty P-stable^r proposition, and:
   - For all $Y \in \mathfrak{A}$ such that $Y \cap X \neq \emptyset$, for all $Z \in \mathfrak{A}$:

$$Bel(Z|Y) \text{ if and only if } Z \supseteq Y \cap X$$

   (and hence, $B_W = X$)

Furthermore, if 1 is the case, then X in 2 is actually uniquely determined.

"Every believed proposition must then have a probability that lies somewhere in the closed interval $[P(X), 1]$ so that $P(X)$ becomes a lower threshold value; furthermore, since X is P-stable^r, $P(X)$ itself is strictly bounded from below by r...r is not necessarily give by the result of applying P to some distinguished proposition or the like—it could be chosen before any considerations on P or *Bel* commence. "

Finding P-stable^r sets

Suppose that W is finite:

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3. If $P(\{w_1\}) > P(\{w_2\}) + \cdots + P(\{w_n\})$ then $\{w_1\}$ is the first P-stable$^{\frac{1}{2}}$ set.

Finding P-stable^r sets

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2. If both $P(\{w_1\})$ and $P(\{w_2\})$ are greater than $P(\{w_3\}) + \cdots + P(\{w_n\})$, then $\{w_1, w_2\}$ is the (next) P-stable$^{\frac{1}{2}}$ set and move onto the list $P(\{w_3\}), \ldots, P(\{w_n\})$.

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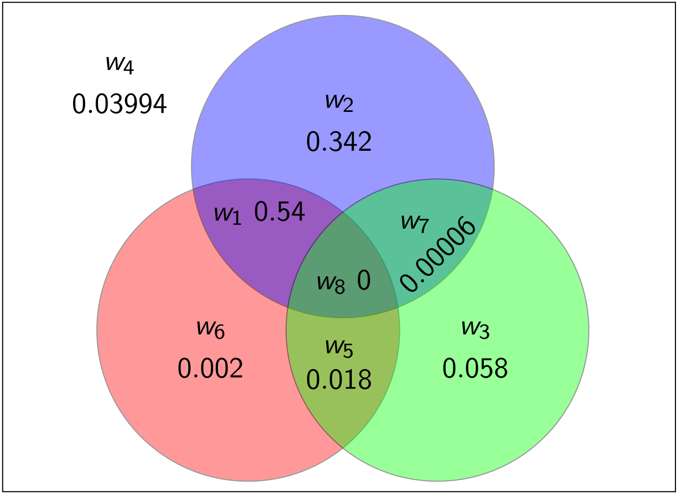
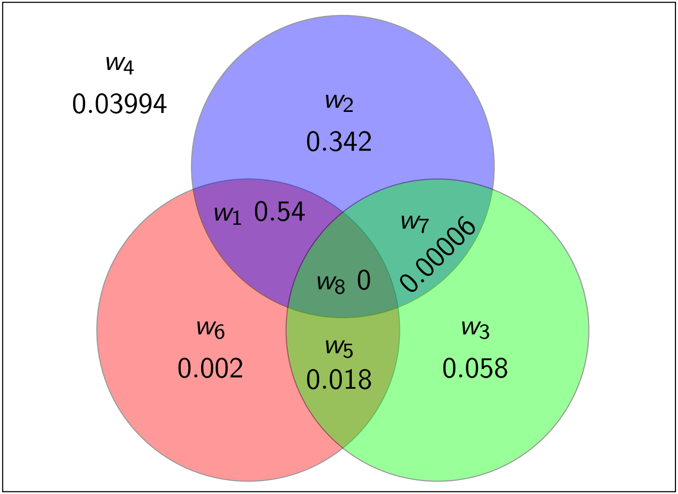
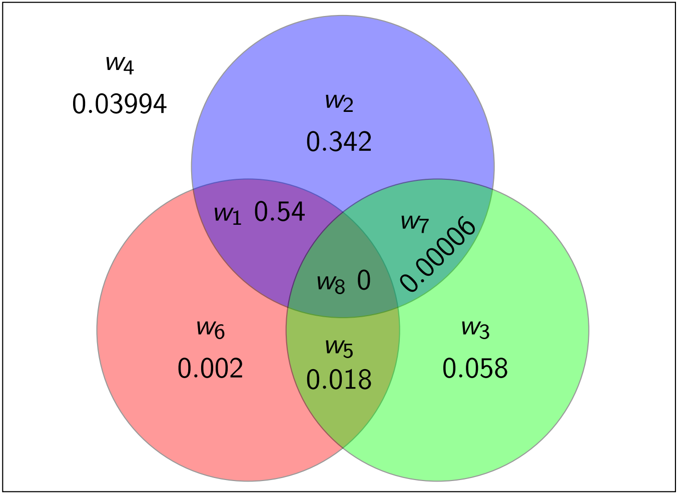
3. If either are less than or equal to $P(\{w_3\}) + \cdots + P(\{w_n\})$ then consider $P(\{w_1\}), P(\{w_2\}), P(\{w_3\})$ and so forth.

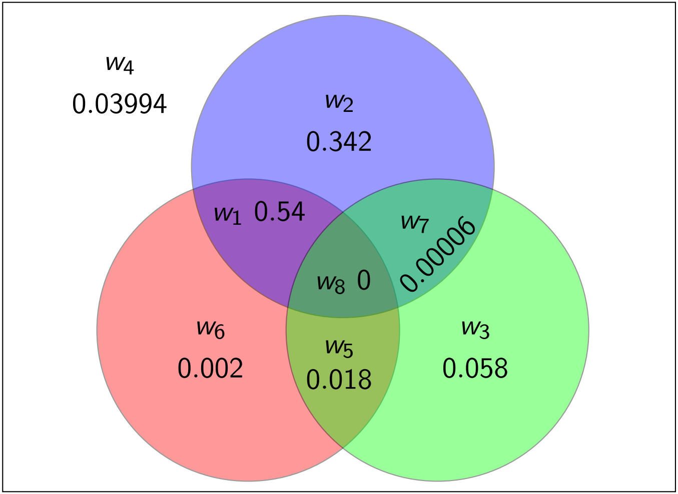

w_4
0.03994
w_2
0.342
w_1 0.54
w_7
0.00006
w_8 0
w_6
0.002
w_5
0.018
w_3
0.058



$\{w_1\}$, $\{w_1, w_2\}$, $\{w_1, w_2, w_3, w_4\}$, ... are P-stable$^{\frac{1}{2}}$



$\{w_1\}$, $\{w_1, w_2\}$, $\{w_1, w_2, w_3, w_4\}$, ... are P-stable$^{\frac{1}{2}}$
neither $\{w_1, w_2, w_3\}$ nor $\{w_1, w_2, w_4\}$ are P-stable$^{\frac{1}{2}}$.



$\{w_1, \ldots, w_5\}, \{w_1, \ldots, w_6\}$ are P-stable$^{\frac{3}{4}}$

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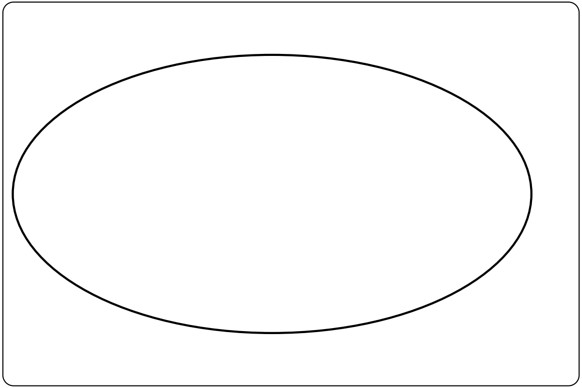
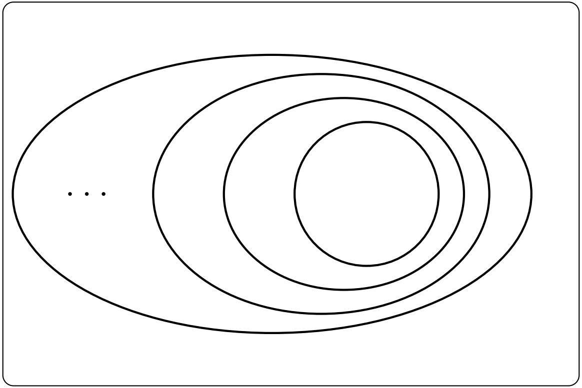
2. If P also satisfies $P2$, then there is no infinitely descending chain of sets in $\mathfrak{A}$ that are all subsets of some P-stable^r set X_0 of $\mathfrak{A}$ with probability less than 1. That is, there is no countably infinite sequence

$$X_0 \supsetneq X_1 X_2 \supsetneq \cdots$$

of sets in $\mathfrak{A}$ such that X_0 is P-stable^r, $P(X_0) < 1$ and each X_n is a proper superset of X_{n+1}.



- With P2 and $r \geq \frac{1}{2}$, The class of P-stable^r propositions X in $\mathfrak{A}$ with $P(X) < 1$ is well-ordered with respect to the subset relation.
- If there is a non-empty P-stable^r $X \in \mathfrak{A}$ with $P(X) < 1$, then there is also a least such X.



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BP2 (Zero Supposition) For all $Y \in \mathfrak{A}$: If $P(Y) = 0$ and $Y \cap B_W \neq \emptyset$, then $B_Y \neq \emptyset$

This implies that there is a least X such that $P(X) = 1$.

BP3 (Maximiality) Among all classes Bel' of ordered pairs of members of $\mathfrak{A}$, such that P and Bel' jointly satisfy $P1 - P2$, $B1 - B6$, $BP1^r$, $BP2$, the class Bel is the *largest* with respect to the class of beliefs.

Let $P : \mathfrak{A} \rightarrow [0,1]$ be a countably additive probability measure on a σ-algebra $\mathfrak{A}$, such that there exists a least set of probability 1 in $\mathfrak{A}$.

Let X_{least} be the least non-empty P-stable^r proposition in $\mathfrak{A}$ (which exists).

Then we say for all $Y \in \mathfrak{A}$ and $\frac{1}{2} \leq r < 1$: $Bel_P^r(Y)$ iff $Y \supseteq X_{least}$ (i.e., Y is believed to a cautiousness degree of r as given by P)

- *Lottery Paradox*: Given a uniform measure P on a finite set W, W is the only P-stable^r set with $r \geq \frac{1}{2}$; so only W is believed.

- *Preface Paradox*: $Bel(X_1), \ldots Bel(X_n)$, $Bel(\neg X_1 \vee \cdots \vee \neg X_n)$ is impossible, but we can have $Bel(X_1), \ldots Bel(X_n)$, $P(\neg X_1 \vee \cdots \vee \neg X_n) > 0$

- If $r < \frac{1}{2}$, then we can take *Bel* to express a weaker epistemic attitude, i.e., Suppose X, proposition Y is an interesting or salient thesis that is to be investigated further.