Proof of Gärdenfors Impossibility Result

Eric Pacuit*

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Let $K/A = Cn(K \cup \{A\})$

Assumptions:

1. A, B and C are three propositions such that
   (a) $\neg A \notin K$
   (b) $\neg B \notin K$
   (c) $\neg C \notin K$
   (d) $\vdash \neg(A \wedge B)$
   (e) $\vdash \neg(A \wedge C)$
   (f) $\vdash \neg(B \wedge C)$

(1) $A \in K_A$

(2) If $K \neq K_\perp$ and $K_A = K_\perp$ then $\vdash \neg A$

(R) $A \;>\; B \in K$ iff $B \in K * A$

(M) For all K, K' and A, $K \subseteq K'$ implies $K_A \subseteq K'_A$

(P) If $\neg A \notin K$ and $B \in K$ then $B \in K_A$

(3) $(K/A)/B = K/(A \wedge B)$
$Cn(Cn(K \cup \{A\}) \cup \{B\}) = Cn(K \cup \{A \wedge B\})$

(4) $K/(A \vee B) \subseteq K/A$
$Cn(K \cup \{A \vee B\}) \subseteq Cn(K \cup \{A\})$

(I) If $\neg A \notin K$ then $K/A \subseteq K_A$

Lemma 0.1 *(R) implies (M).*

Proof. Suppose that $K \subseteq K'$ and $B \in K_A$. By (R), $A \;>\; B \in K$ which implies $A \;>\; B \in K'$. Again by (R), $B \in K'_A$. Since B was arbitrary, $K_A \subseteq K'_A$. QED

* *Webpage*: pacuit.org, *Email*: epacuit@umd.edu

Theorem 0.2 *There is no non-trivial belief revision model that satisfies all the conditions (1), (2), (M) and (P).*

Proof.

- Suppose A, B and C satisfy 1. above.
- We first show that $\neg C \notin (K/(A \vee B))_{B \vee C}$
  1. By (1), $B \vee C \in (K_A)_{B \vee C}$
  2. Since, $(K_A)_{B \vee C}$ is consistent (this follows from (2)), either $\neg B \notin (K_A)_{B \vee C}$ or $\neg C \notin (K_A)_{B \vee C}$.
  3. Assume $\neg C \notin (K_A)_{B \vee C}$ (the other case is similar).
  4. By (4), $K/(A \vee B) \subseteq K/A$
  5. Since $\neg A \notin K$ (assumption 1(a)), by (I) $K/(A \vee B) \subseteq K/A \subseteq K_A$.
  6. By (M), $(K/(A \vee B))_{B \vee C} \subseteq (K_A)_{B \vee C}$
  7. Hence, $\neg C \notin (K/(A \vee B))_{B \vee C}$ (from 3.)
- Now show that $\neg C \in (K/(A \vee B))_{B \vee C}$
  1. $\neg(B \vee C) \notin K/(A \vee B)$. (Suppose $\neg(B \vee C) \in K/(A \vee B)$. By (4), $K/(A \vee B) \subseteq K/B$. Hence $\neg(B \vee C) \in K/B$. This is a contradiction, since it implies $\neg B \in K/B$ and of course $B \in K/B$.)
  2. By (I), $(K/(A \vee B))/(B \vee C) \subseteq (K/(A \vee B))_{B \vee C}$.
  3. By (3), $(K/(A \vee B))/(B \vee C) = K/((A \vee B) \wedge (B \vee C))$. Since A, B and C are pairwise disjoint, we have $K/((A \vee B) \wedge (B \vee C)) = K/B$.
  4. By 2. and 3.,$K/B \subseteq (K/(A \vee B))_{B \vee C}$.
  5. Since $B \in K/B$ and $\vdash \neg(B \wedge C)$, we have $\neg C \in K/B$.
  6. Hence, $\neg C \in (K/(A \vee B))_{B \vee C}$.
- This is a contradiction.

QED