Reasoning about Knowledge and Beliefs

Lecture 11

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Robert Aumann. *Agreeing to Disagree*. Annals of Statistics **4** (1976).

"A group of agents cannot agree to disagree"

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Theorem. Suppose that n agents share a common prior and have different private information. If there is common knowledge in the group of the posterior probabilities, then the posteriors must be equal.

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It is not Common Knowledge who "defined" Common Knowledge!

The first formal definition of common knowledge?

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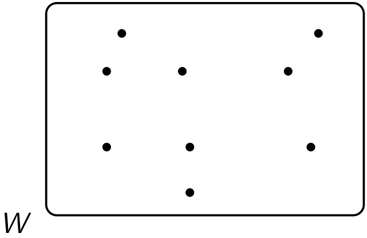
Shared situation: There is a *shared situation* s such that (1) s entails φ, (2) s entails everyone knows φ, plus other conditions

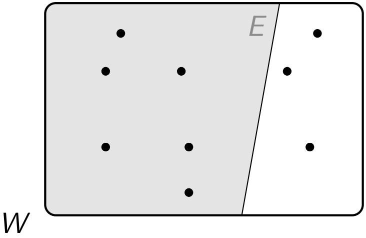
H. Clark and C. Marshall. *Definite Reference and Mutual Knowledge*. 1981.

M. Gilbert. *On Social Facts*. Princeton University Press (1989).

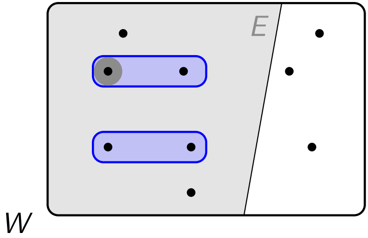
P. Vanderschraaf and G. Sillari. *"Common Knowledge", The Stanford Encyclopedia of Philosophy (2009)*.
http://plato.stanford.edu/entries/common-knowledge/.



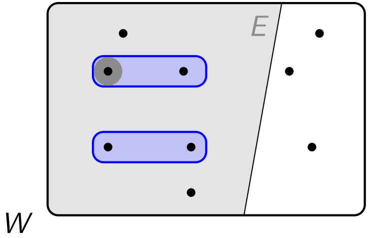
W is a set of **states** or **worlds**.



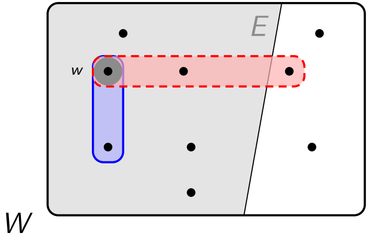
An **event**/**proposition** is any (definable) subset $E \subseteq W$



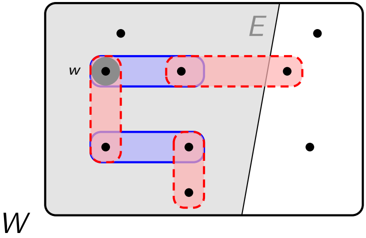
The agents receive signals in each state. States are considered equivalent for the agent if they receive the same signal in both states.



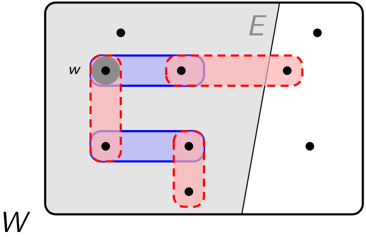
Knowledge Function: $K_i : \wp(W) \to \wp(W)$ where $K_i(E) = \{w \mid R_i(w) \subseteq E\}$



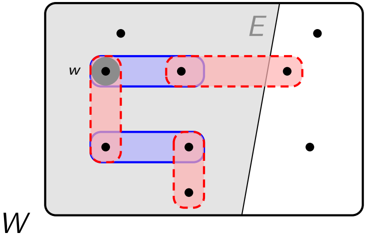
$w \in K_A(E)$ and $w \notin K_B(E)$



The model also describes the agents' **higher-order knowledge/beliefs**

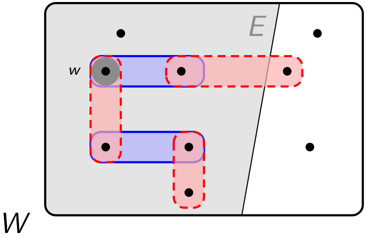


Everyone Knows: $K(E) = \bigcap_{i\in\mathcal{A}} K_i(E)$, $K^0(E) = E$, $K^m(E) = K(K^{m-1}(E))$

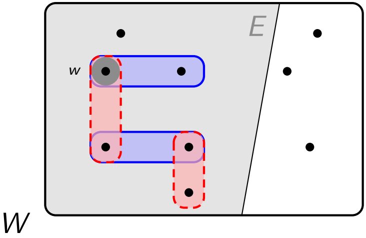


Common Knowledge: $C : \wp(W) \to \wp(W)$ with

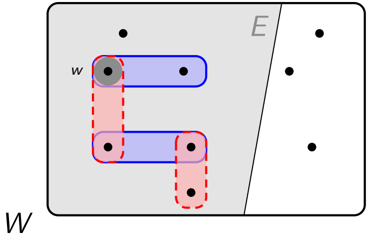
$$C(E) = \bigcap_{m \geq 0} K^m(E)$$



$w \in K(E)$ $\qquad$ $w \notin C(E)$



$w \in C(E)$



Fact. $w \in C(E)$ if every finite path starting at w ends in a state in E

An Example

Two players Ann and Bob are told that the following will happen. Some positive integer n will be chosen and *one* of n, $n + 1$ will be written on Ann's forehead, the other on Bob's. Each will be able to see the other's forehead, but not his/her own.

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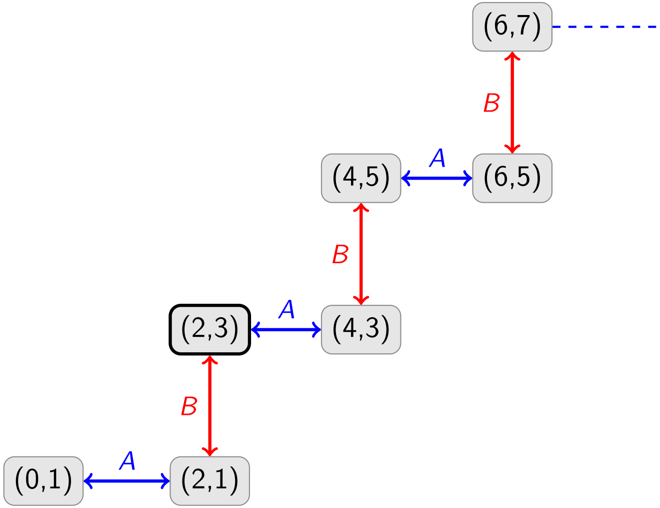
Is it common knowledge that their numbers are less than 1000?


(6,7)
B
(4,5)
A
(6,5)
B
(2,3)
A
(4,3)
B
(0,1)
A
(2,1)

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Suppose you are told "Ann and Bob are going together,"' and respond "sure, that's common knowledge." What you mean is not only that everyone knows this, but also that the announcement is pointless, occasions no surprise, reveals nothing new; in effect, that the situation after the announcement does not differ from that before. ...the event "Ann and Bob are going together" — call it E — is common knowledge if and only if some event — call it F — happened that entails E and also entails all players' knowing F (like all players met Ann and Bob at an intimate party). *(Aumann, pg. 271, footnote 8)*

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The following axiomatize common knowledge:

- $C(\varphi \to \psi) \to (C\varphi \to C\psi)$
- $C\varphi \to (\varphi \wedge EC\varphi)$ (Fixed-Point)
- $C(\varphi \to E\varphi) \to (\varphi \to C\varphi)$ (Induction)

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- (Tarski) Every monotone operator has a greatest (and least) fixed point
- Let $K^*(E)$ be the greatest fixed point of f_E.
- **Fact**. $K^*(E) = C(E)$.

The Fixed-Point Definition

Separating the fixed-point/iteration definition of common knowledge/belief:

J. Barwise. *Three views of Common Knowledge*. TARK (1987).

J. van Benthem and D. Saraenac. *The Geometry of Knowledge*. Aspects of Universal Logic (2004).

A. Heifetz. *Iterative and Fixed Point Common Belief*. Journal of Philosophical Logic (1999).

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- $D_G(\varphi) \to \bigwedge_{i \in G} K_i \varphi$

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$$w \in K_G(E) \text{ iff } R_G(w) \subseteq E \quad (\text{without necessarily } R_G(w) = \bigcap_{i \in G} R_i(w))$$

A. Baltag and S. Smets. *Correlated Knowledge: an Epistemic-Logic view on Quantum Entanglement*. Int. Journal of Theoretical Physics (2010).

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S. Morris. *The common prior assumption in economic theory*. Economics and Philosophy, 11, pgs. 227 - 254, 1995.

Generalized Aumann's Theorem

Qualitative versions: *like-minded individuals cannot agree to make different decisions*.

M. Bacharach. *Some Extensions of a Claim of Aumann in an Axiomatic Model of Knowledge*. Journal of Economic Theory (1985).

J.A.K. Cave. *Learning to Agree*. Economic Letters (1983).

D. Samet. *Agreeing to disagree: The non-probabilistic case*. Games and Economic Behavior, Vol. 69, 2010, 169-174.

The Framework

Knowledge Structure: $\langle W, \{\Pi_i\}_{i \in \mathcal{A}} \rangle$ where each Π_i is a partition on W ($\Pi_i(w)$ is the cell in Π_i containing w).

Decision Function: Let D be a nonempty set of **decisions**. A decision function for $i \in \mathcal{A}$ is a function $\mathbf{d}_i : W \to D$. A vector $\mathbf{d} = (d_1, \ldots, d_n)$ is a decision function profile. Let $[\mathbf{d}_i = d] = \{w \mid \mathbf{d}_i(w) = d\}$.

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(A1) Each agent knows her own decision:

$$[\mathbf{d}_i = d] \subseteq K_i([\mathbf{d}_i = d])$$

Comparing Knowledge

$[j \succeq i]$: agent j is at least as knowledgeable as agent i.

$$[j \succeq i] := \bigcap_{E \in \wp(W)} (K_i(E) \Rightarrow K_j(E)) = \bigcap_{E \in \wp(W)} (\neg K_i(E) \cup K_j(E))$$

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$[j \sim i] = [j \succeq i] \cap [i \succeq j]$

The Sure-Thing Principle

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Sure-Thing Principle

Should I study or have a beer?

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Should I study or have a beer? Either I pass or I won't pass the exam. If I pass, it is better to drink and pass, so I should drink. If I fail, it is better to drink and fail, so I should drink.

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Sure-Thing Principle

There are three candidates, republican, independent and democrat.

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There are three candidates, republican, independent and democrat. I will buy stock if the democrat looses and I will buy stock if the republican looses.

Sure-Thing Principle

There are three candidates, republican, independent and democrat. I will buy stock if the democrat looses and I will buy stock if the republican looses. Either the republican or democrat will loose. So, I should buy the stock.

R. Aumann, S. Hart and M. Perry. *Conditioning and the Sure-Thing Principle*. manuscript, 2005.

The Nixon Diamond

You're told (from a reliable source) that Nixon is a republican, which suggests that he is a Hawk. You're also told (from a reliable source) that Nixon is a Quaker, which suggests that he is a Dove.

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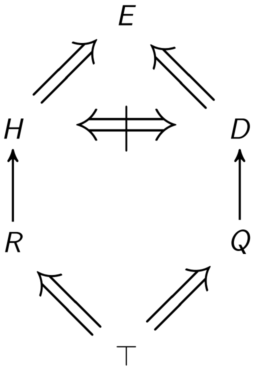
Floating Conclusions



J. Horty. *Skepticism and floating conclusions*. Artificial Intelligence, 135, pp. 55 - 72, 2002.

Your parents have 1M inheritance which will is split between you mother and father (each may give you 0.5M).

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Interpersonal Sure-Thing Principle (ISTP)

For any pair of agents i and j and decision d,

$$K_i([j \succeq i] \cap [\mathbf{d}_j = d]) \subseteq [\mathbf{d}_i = d]$$

Interpersonal Sure-Thing Principle (ISTP): Illustration

Suppose that Alice and Bob, two detectives who graduated the same police academy, are assigned to investigate a murder case.

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Interpersonal Sure-Thing Principle (ISTP): Illustration

Suppose that Alice and Bob, two detectives who graduated the same police academy, are assigned to investigate a murder case. If they are exposed to different evidence, they may reach different decisions. Yet, being the students of the same academy, the method by which they arrive at their conclusions is the same. Suppose now that detective Bob, a father of four who returns home every day at five oclock, collects all the information about the case at hand together with detective Alice.

Interpersonal Sure-Thing Principle (ISTP): Illustration

However, Alice, single and a workaholic, continues to collect more information every day until the wee hours of the morning — information which she does not necessarily share with Bob.

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However, Alice, single and a workaholic, continues to collect more information every day until the wee hours of the morning — information which she does not necessarily share with Bob. Obviously, Bob knows that Alice is at least as knowledgeable as he is. Suppose that he also knows what Alices decision is. Since Alice uses the same investigation method as Bob, he knows that had he been in possession of the more extensive knowledge that Alice has collected, he would have made the same decision as she did. Thus, this is indeed his decision.

Implications of ISTP

Proposition. If the decision function profile **d** satisfies ISTP, then

$$[i \sim j] \subseteq \bigcup_{d \in D} ([\mathbf{d}_i = d] \cap [\mathbf{d}_j = d])$$

ISTP Expandability

Agent i is an **epistemic dummy** if it is always the case that all the agents are at least as knowledgeable as i. That is, for each agent j,

$$[j \succeq i] = W$$

A decision function profile $\mathbf{d}$ on $\langle W, \Pi_1, \ldots, \Pi_n \rangle$ is **ISTP expandable** if for any expanded structure $\langle W, \Pi_1, \ldots, \Pi_{n+1} \rangle$ where $n+1$ is an epistemic dummy, there exists a decision function $\mathbf{d}_{n+1}$ such that $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_{n+1})$ satisfies ISTP.

ISTP Expandability: Illustration

Suppose that after making their decisions, Alice and Bob are told that another detective, one E.P. Dummy, who graduated the very same police academy, had also been assigned to investigate the same case.

ISTP Expandability: Illustration

Suppose that after making their decisions, Alice and Bob are told that another detective, one E.P. Dummy, who graduated the very same police academy, had also been assigned to investigate the same case. In principle, they would need to review their decisions in light of the third detectives knowledge: knowing what they know about the third detective, his usual sources of information, for example, may impinge upon their decision.

ISTP Expandability: Illustration

But this is not so in the case of detective Dummy. It is commonly known that the only information source of this detective, known among his colleagues as the couch detective, is the TV set.

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But this is not so in the case of detective Dummy. It is commonly known that the only information source of this detective, known among his colleagues as the couch detective, is the TV set. Thus, it is commonly known that every detective is at least as knowledgeable as Dummy. The news that he had been assigned to the same case is completely irrelevant to the conclusions that Alice and Bob have reached. Obviously, based on the information he gets from the media, Dummy also makes a decision.

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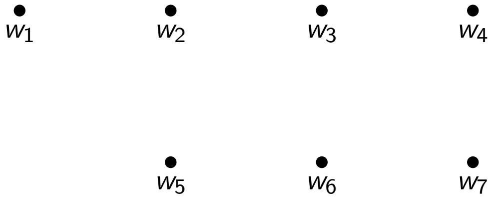
Generalized Agreement Theorem

If $\mathbf{d}$ is an ISTP expandable decision function profile on a partition structure $\langle W, \Pi_1, \ldots, \Pi_n \rangle$, then for any decisions $d_1, \ldots, d_n$ which are not identical, $C(\bigcap_i [\mathbf{d}_i = d_i]) = \emptyset$.

Robert Aumann. *Agreeing to Disagree*. Annals of Statistics **4** (1976).

Theorem. Suppose that n agents share a common prior and have different private information. If there is common knowledge in the group of the posterior probabilities, then the posteriors must be equal.

2 Scientists Perform an Experiment



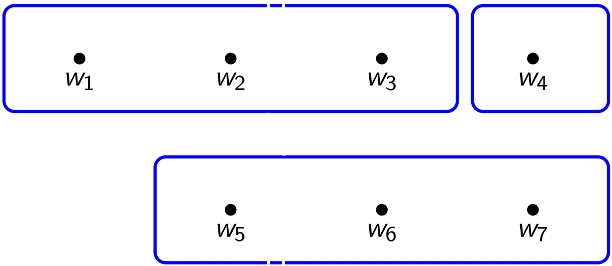
They agree the true state is one of seven different states.

2 Scientists Perform an Experiment

$\frac{2}{32}$ $\bullet$ w_1	$\frac{4}{32}$ $\bullet$ w_2	$\frac{8}{32}$ $\bullet$ w_3	$\frac{4}{32}$ $\bullet$ w_4
	$\frac{5}{32}$ $\bullet$ w_5	$\frac{7}{32}$ $\bullet$ w_6	$\frac{2}{32}$ $\bullet$ w_7

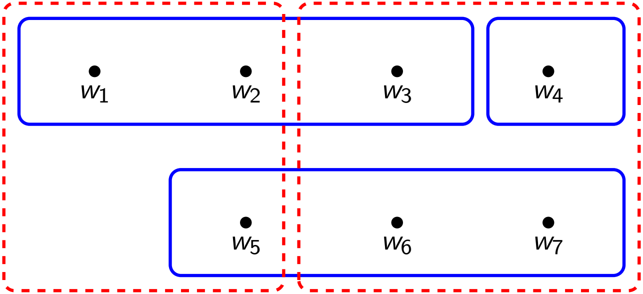
They agree on a common prior.

2 Scientists Perform an Experiment



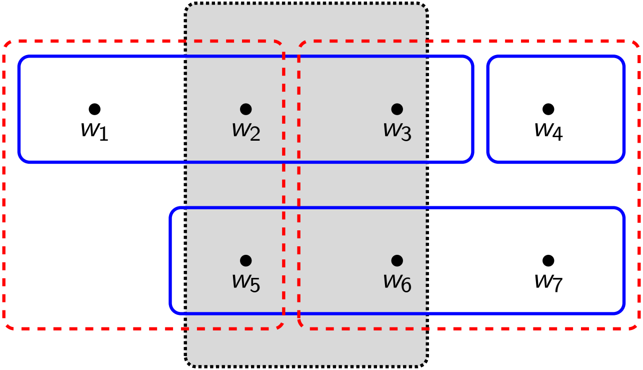
They agree that Experiment 1 would produce the blue partition.

2 Scientists Perform an Experiment



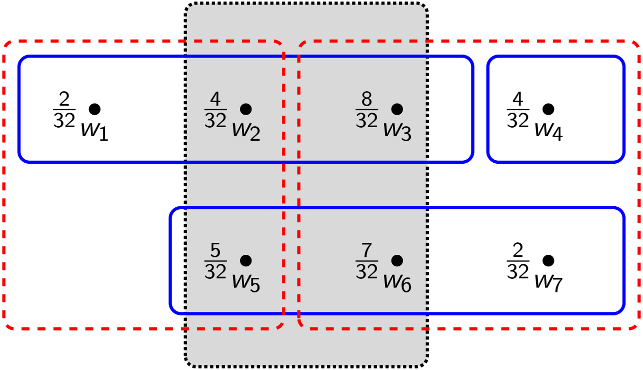
They agree that Experiment 1 would produce the blue partition and Experiment 2 the red partition.

2 Scientists Perform an Experiment



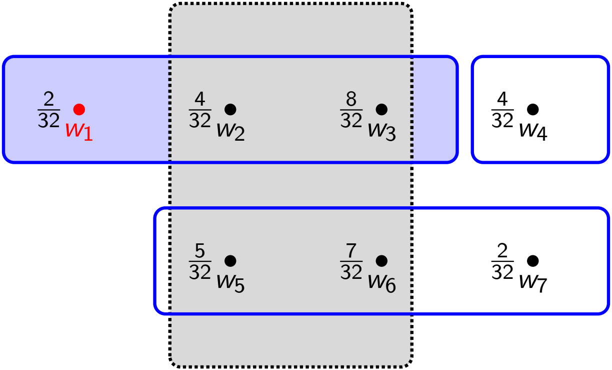
They are interested in the truth of $E = \{w_2, w_3, w_5, w_6\}$.

2 Scientists Perform an Experiment



So, they agree that $P(E) = \frac{24}{32}$.

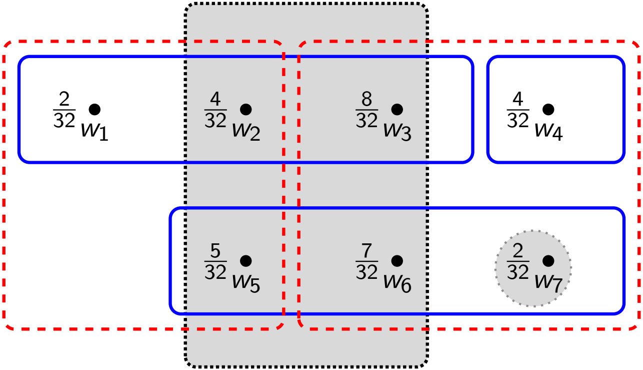
2 Scientists Perform an Experiment



Also, that *if the true state is* w_1, then Experiment 1 will yield

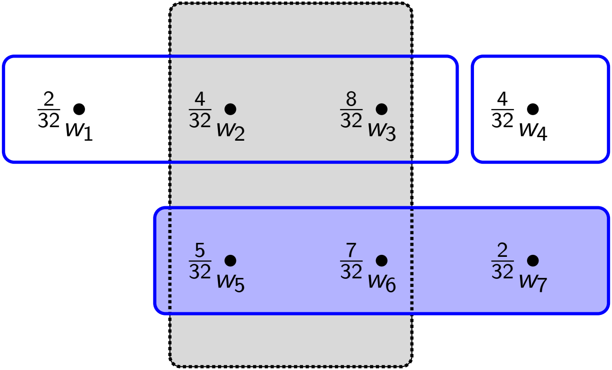
$$P(E|I) = \frac{P(E \cap I)}{P(I)} = \frac{12}{14}$$

2 Scientists Perform an Experiment



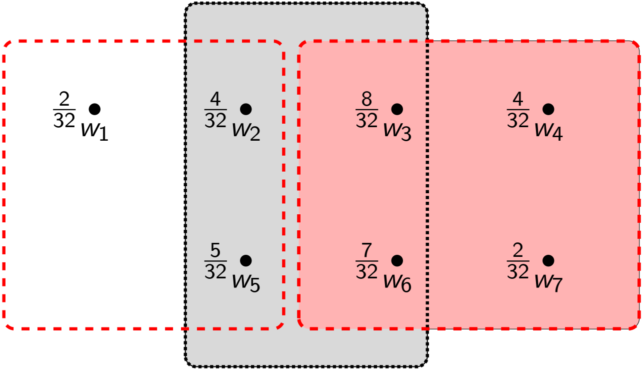
Suppose the true state is w_7 and the agents preform the experiments.

2 Scientists Perform an Experiment



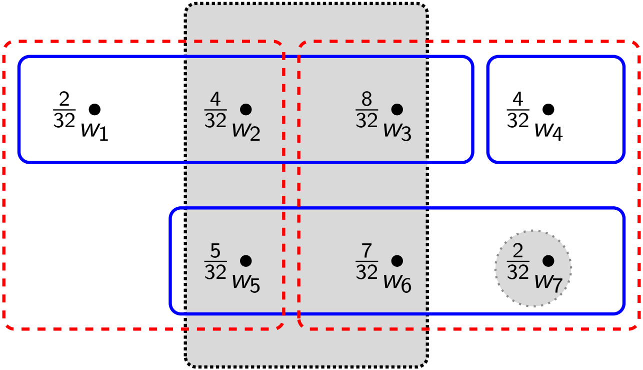
Suppose the true state is w_7, then $Pr_1(E) = \frac{12}{14}$

2 Scientists Perform an Experiment



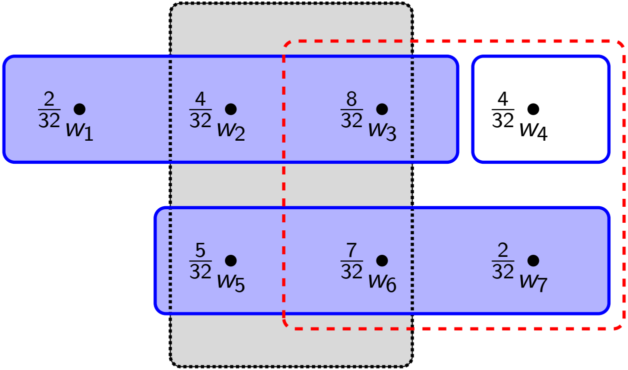
Then $Pr_1(E) = \frac{12}{14}$ and $Pr_2(E) = \frac{15}{21}$

2 Scientists Perform an Experiment



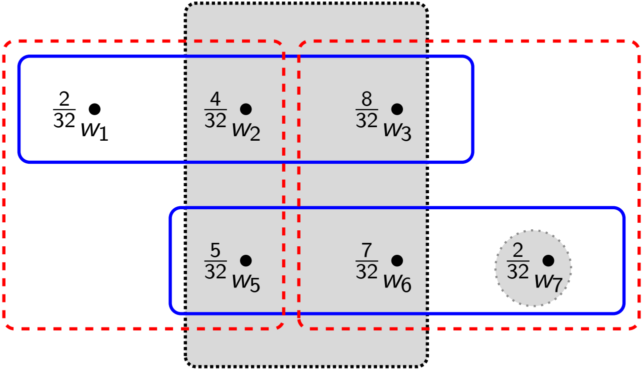
Suppose they exchange emails with the new subjective probabilities: $Pr_1(E) = \frac{12}{14}$ and $Pr_2(E) = \frac{15}{21}$

2 Scientists Perform an Experiment



Agent 2 learns that w_4 is **NOT** the true state (same for Agent 1).

2 Scientists Perform an Experiment



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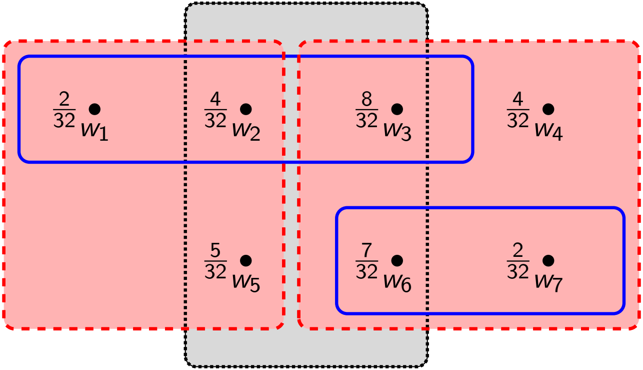
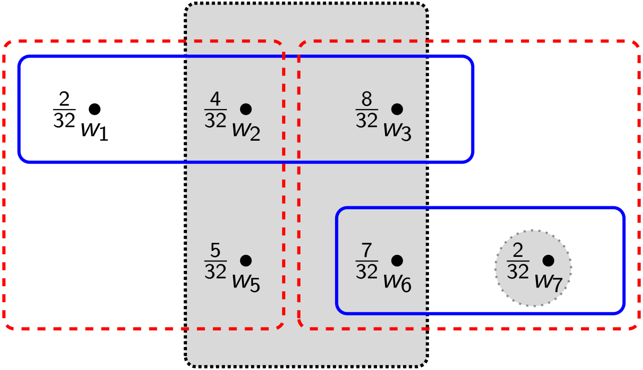


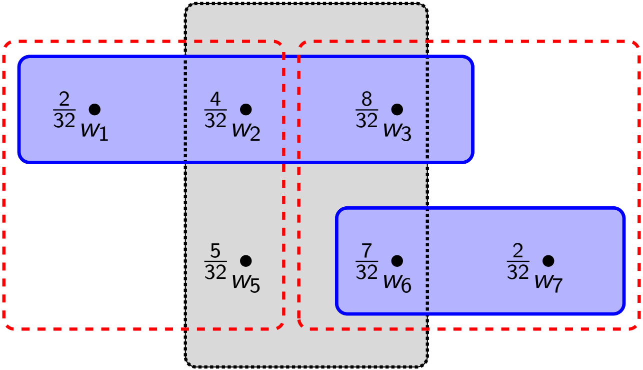
Agent 1 learns that w_5 is **NOT** the true state (same for Agent 1).

2 Scientists Perform an Experiment



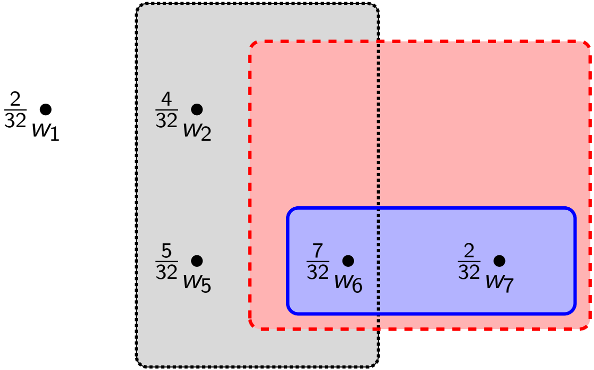
The new probabilities are $Pr_1(E|I') = \frac{7}{9}$ and $Pr_2(E|I') = \frac{15}{17}$

2 Scientists Perform an Experiment



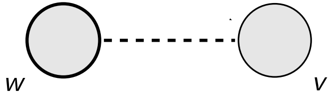
After exchanging this information ($Pr_1(E|I') = \frac{7}{9}$ and $Pr_2(E|I') = \frac{15}{17}$), Agent 2 learns that w_3 is **NOT** the true state.

2 Scientists Perform an Experiment



No more revisions are possible and the agents agree on the posterior probabilities.

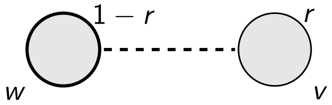
Models of Hard and Soft Information



$\mathcal{M} = \langle W, \{\Pi_i\}_{i \in \mathcal{A}} \rangle$
Π_i is agent i's partition with $\Pi_i(w)$ the partition cell containing w.

$K_i(E) = \{w \mid \Pi_i(w) \subseteq E\}$

Models of Hard and Soft Information



$\mathcal{M} = \langle W, \{\Pi_i\}_{i \in \mathcal{A}}, \{p_i\}_{i \in \mathcal{A}} \rangle$
for each i, $p_i : W \to [0, 1]$ is a probability measure

$B^p(E) = \{w \mid p_i(E \mid \Pi_i(w)) = \frac{\pi_i(E \cap \Pi_i(w))}{p_i(\Pi_i(w))} \geq p\}$

1. $B_i^p(B_i^p(E)) = B_i^p(E)$

2. If $E \subseteq F$ then $B_i^p(E) \subseteq B_i^p(F)$

3. $\pi(E \mid B_i^p(E)) \geq p$

Common p-belief

The typical example of an event that creates common knowledge is a **public announcement**.

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Shouldn't one always allow for some small probability that a participant was absentminded, not listening, sending a text, checking facebook, proving a theorem, asleep, ...

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"We show that the weaker concept of "common belief" can function successfully as a substitute for common knowledge in the theory of equilibrium of Bayesian games."

D. Monderer and D. Samet. *Approximating Common Knowledge with Common Beliefs*. Games and Economic Behavior (1989).

Common p-belief: definition

$$B_i^p(E) = \{w \mid p(E \mid R_i(w)) \geq p\}$$

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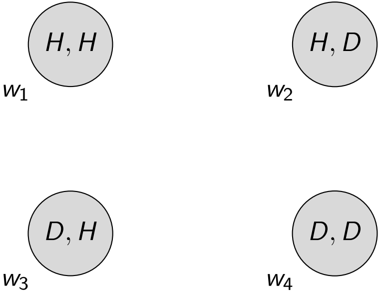
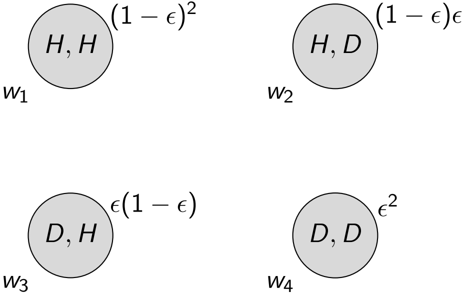
An event F is **common p-belief** at w if there exists and evident p-belief event E such that $w \in E$ and for all $i \in \mathcal{A}$, $E \subseteq B_i^p(F)$

Common p-belief: example



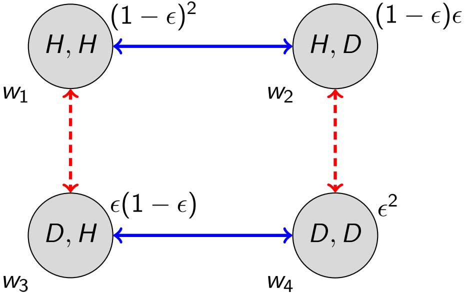
Two agents either hear (H) or don't hear (D) the announcement.

Common p-belief: example



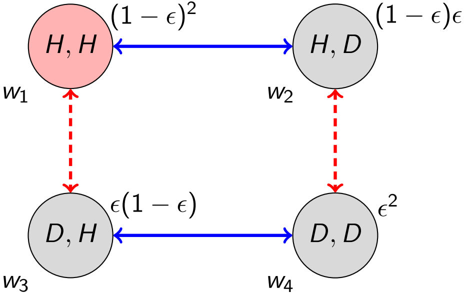
The probability that an agent hears is $1 - \epsilon$.

Common p-belief: example



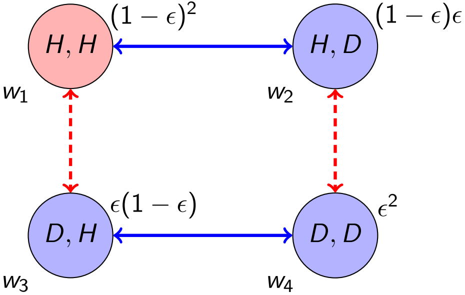
The agents *know* their "type".

Common p-belief: example



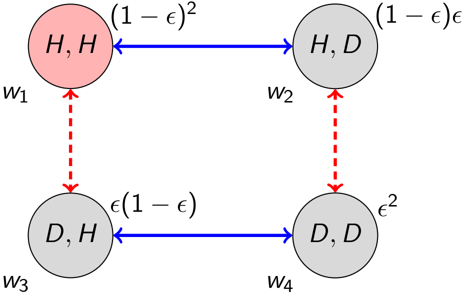
The event "everyone hears" ($E = \{w_1\}$)

Common p-belief: example



The event "everyone hears" ($E = \{w_1\}$) is **not** common knowledge

Common p-belief: example



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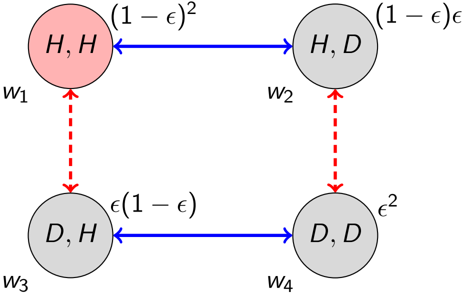
Common p-belief: example



The event "everyone hears" ($E = \{w_1\}$) is **not** common knowledge, but it is common $(1-\epsilon)$-belief:
$B_i^{(1-\epsilon)}(E) = \{w \mid p(E \mid \Pi_i(w)) \geq 1-\epsilon\} = \{w_1\} = E$, for $i = 1, 2$

Common p-belief

Theorem. If the posteriors of an event X are common p-belief at some state w, then any two posteriors can differ by at most $2(1 - p)$.

D. Samet and D. Monderer. *Approximating Common Knowledge with Common Beliefs*. Games and Economic Behavior, Vol. 1, No. 2, 1989.