Reasoning about Knowledge and Beliefs

Lecture 6

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Summary

(Multi-agent) **S5** is a logic of “knowledge”

(Multi-agent) **KD45** is a logic of “belief”

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Two issues:

- Modeling awareness/unawareness
- Logics with both knowledge and belief operators

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- The agent may believe φ and ruled-out the $\neg\varphi$-worlds, but this was based on “bad” evidence, or was not justified, or the agent was “epistemically lucky” (e.g., Gettier cases),...
- The agent has not yet entertained possibilities relevant to the truth of φ (the agent is unaware of φ).

Can we model unawareness in state-space models?

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E. Dekel, B. Lipman and A. Rustichini. *Standard State-Space Models Preclude Unawareness*. Econometrica, 55:1, pp. 159 - 173 (1998).

Properties of Unawareness

1. $U\varphi \rightarrow (\neg K\varphi \wedge \neg K \neg K\varphi)$

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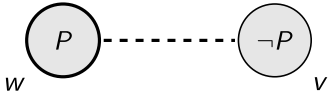
Theorem. In any logic where U satisfies the above axiom schemes, we have

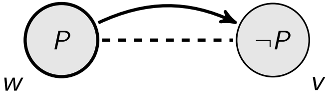
1. If K satisfies Necessitation (from φ infer $K\varphi$), then for all formulas φ, $\neg U\varphi$ is derivable (the agent is aware of everything); and
2. If K satisfies Monotonicity (from $\varphi \to \psi$ infer $K\varphi \to K\psi$), then for all φ and ψ, $U\varphi \to \neg K\psi$ is derivable (if the agent is unaware of something then the agent does not know anything).

B. Schipper. *Online Bibliography on Models of Unawareness*. http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm.

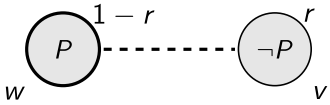
J. Halpern. *Alternative semantics for unawareness*. Games and Economic Behavior, 37, 321-339, 2001.



Ann does not know that P



Ann does not know that P, but she believes that $\neg P$



Ann does not know that P, but she believes that $\neg P$ is true to degree r.

Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim_i$ is an equivalence relation on W;
- each R_i is a serial, transitive, Euclidean relation on W; and
- V is a valuation function.

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- $B_i\varphi \to B_iK_i\varphi$? "positive certainty"

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- $K_i\varphi \rightarrow B_i\varphi$? "knowledge implies belief"
- $B_i\varphi \rightarrow B_iK_i\varphi$? "positive certainty"
- $B_i\varphi \rightarrow K_iB_i\varphi$? "strong introspection"

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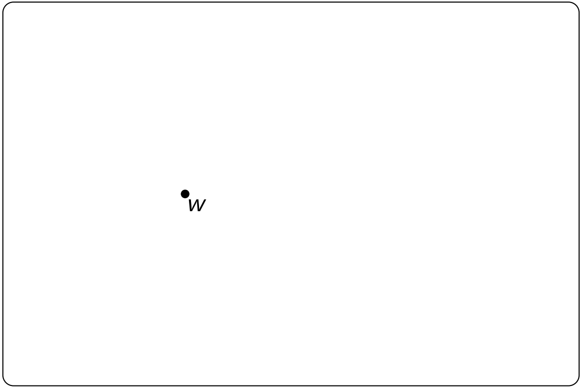
- So, $BKp \wedge B\neg Kp$ also holds, but this contradictions $B\varphi \rightarrow \neg B\neg\varphi$.

J. Halpern. *Should Knowledge Entail Belief?*. Journal of Philosophical Logic, 25:5, 1996, pp. 483-494.



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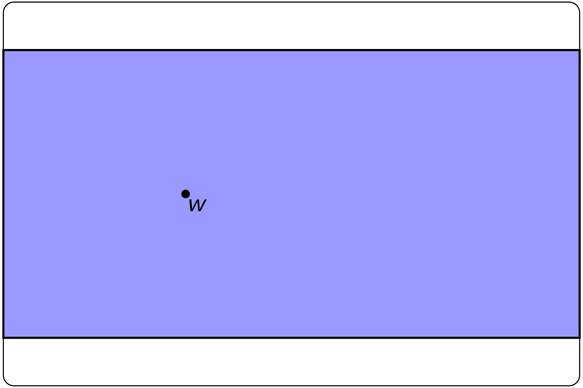

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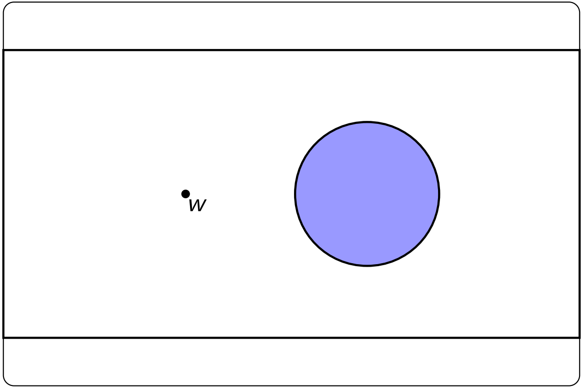
- The agent's (hard) information (i.e., the states consistent with what the agent knows)



- The agent's (hard) information (i.e., the states consistent with what the agent knows)
- The agent's beliefs (soft information—-the states consistent with what the agent believes)

Digression on Belief Change, I

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Consider the following beliefs of a rational agent:

- p_1 All Europeans swans are white.
- p_2 The bird caught in the trap is a swan.
- p_3 The bird caught in the trap comes from Sweden.
- p_4 Sweden is part of Europe.

Thus, the agent believes:

- q The bird caught in the trap is white.

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Now suppose the rational agent—for example, You—learn that the bird caught in the trap is black ($\neg q$).

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There are several logically consistent ways to incorporate $\neg q$!

Digression on Belief Change, II

What extralogical factors serve to determine what beliefs to give up and what beliefs to retain?

Digression on Belief Change, III

Belief revision is a matter of choice, and the choices are to be made in such a way that:

1. The resulting theory squares with the experience;
2. It is simple; and
3. The choices disturb the original theory as little as possible.

Digression on Belief Change, III

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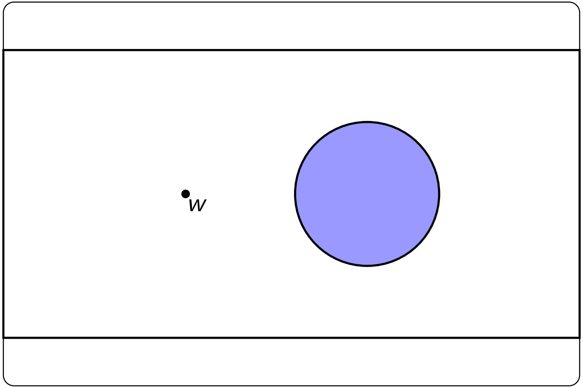
Research has relied on the following related guiding ideas:

1. When accepting a new piece of information, an agent should aim at a minimal change of his old beliefs.
2. If there are different ways to effect a belief change, the agent should give up those beliefs which are least entrenched.

Digression: Belief Revision

A.P. Pedersen and H. Arló-Costa. *"Belief Revision."*. In Continuum Companion to Philosophical Logic. Continuum Press, 2011.

Hans Rott. *Change, Choice and Inference: A Study of Belief Revision and Nonmonotonic Reasoning*. Oxford University Press, 2001.



- The agent's (hard) information (i.e., the states consistent with what the agent knows)
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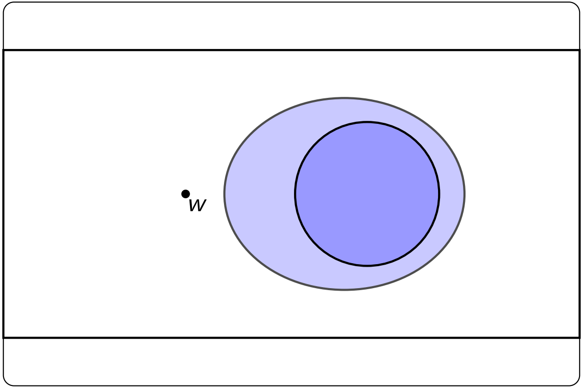

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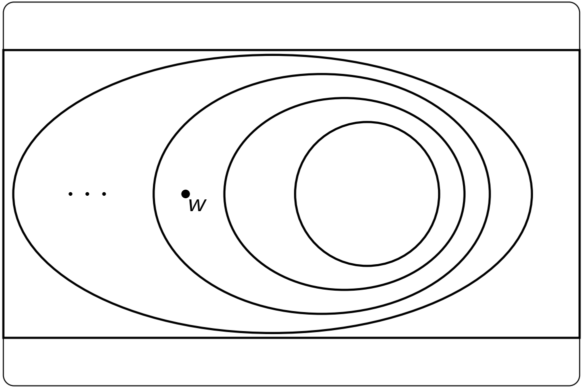

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Sphere Models

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Let W be a set of states, A system of spheres $\mathcal{F} \subseteq \wp(W)$ such that:

- For each $S, S' \in \mathcal{F}$, either $S \subseteq S'$ or $S' \subseteq S$
- For any $P \subseteq W$ there is a smallest $S \in \mathcal{F}$ (according to the subset relation) such that $P \cap S \neq \emptyset$
- The spheres are non-empty $\bigcap \mathcal{F} \neq \emptyset$ and cover the entire information cell $\bigcup \mathcal{F} = W$ (or $[w] = \{v \mid w \sim v\}$)

Let $\mathcal{F}$ be a system of spheres on W: for $w, v \in W$, let

$$w \preceq_{\mathcal{F}} v \text{ iff for all } S \in \mathcal{F}, \text{ if } v \in S \text{ then } w \in S$$

Then, $\preceq_{\mathcal{F}}$ is reflexive, transitive, and well-founded.

$w \preceq_{\mathcal{F}} v$ means that: no matter what the agent learns in the future, as long as world v is still consistent with his beliefs and w is still epistemically possible, then w is also consistent with his beliefs.

Plausibility Models

Epistemic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, V \rangle$

Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \mathsf{At}$)
- $\mathcal{M}, w \models \neg\varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K_i\varphi$ if for each $v \in W$, if $w \sim_i v$, then $\mathcal{M}, v \models \varphi$

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Epistemic-Plausibility Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{\preceq_i\}_{i \in \mathcal{A}}, V \rangle$

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"w is at least as plausible as v."

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Most Plausible: For $X \subseteq W$, let

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Assumptions:

1. *plausibility implies possibility*: if $w \preceq_i v$ then $w \sim_i v$.
2. *locally-connected*: if $w \sim_i v$ then either $w \preceq_i v$ or $v \preceq_i w$.

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- $\mathcal{M}, w \models K_i\varphi$ if for each $v \in W$, if $w \sim_i v$, then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B_i\varphi$ if for each $v \in Min_{\preceq_i}([w]_i)$, $\mathcal{M}, v \models \varphi$
  $[w]_i = \{v \mid w \sim_i v\}$ is the agent's **information cell**.