Reasoning about Knowledge and Beliefs

Lecture 5

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Taking Stock

Multi-agent language: $\varphi \ := \ p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i\varphi$

- $\Box_i\varphi$: "agent i knows that φ" (write $K_i\varphi$ for $\Box_i\varphi$)
- $\Box_i\varphi$: "agent i believes that φ" (write $B_i\varphi$ for $\Box_i\varphi$)

Kripke Models: $\mathcal{M} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$

Truth: $\mathcal{M}, w \models \Box_i\varphi$ iff for all $v \in W$, if wR_iv then $\mathcal{M}, v \models \varphi$

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$\neg\Box\bot$	Serial

The Logic **S5**

The logic **S5** contains the following axioms and rules:

Pc	Axiomatization of Propositional Calculus
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
5	$\neg K\varphi \to K\neg K\varphi$
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Theorem

S5 *is sound and strongly complete with respect to the class of Kripke frames with equivalence relations.*

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Theorem

KD45 *is sound and strongly complete with respect to the class of Kripke frames with pseudo-equivalence relations (reflexive, transitive and serial).*

Truth Axiom/Consistency

$$K\varphi \to \varphi$$

$$\neg B\bot$$

Negative Introspection

$$\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$$

$(\Box = K, B)$

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- The agent has not yet entertained possibilities relevant to the truth of φ (the agent is unaware of φ).

Positive Introspection

$$\Box\varphi \to \Box\Box\varphi$$

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The KK Principle

More famous is the "KK principle" (or "positive introspection"):

$$4_i \quad K_i\varphi \to K_iK_i\varphi.$$

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How Many Modalities?

Fact. In **S5** and **KD45**, there are only three modalities ($\Box$, $\Diamond$, and the "empty modality")

Williamson's Margin of Error Puzzle

We will now consider an argument, due to Williamson, that purports to be a *reductio ad absurdum* of the KK principle.

T. Williamson. 2000. *Knowlege and Its Limits*, Oxford University Press

T. Williamson. 2007. "Rational Failures of the KK Principle."
The Logic of Strategy, eds. C. Bicchieri, R. Jeffrey, and B. Skyrms, OUP.

Williamson's Margin of Error Puzzle

Suppose an agent is estimating the height of a faraway tree, which is in fact k inches. While the agent's rationality is perfect, his eyesight is not. As Williamson (2000) explains, "anyone who can tell by looking at the tree that it is not i inches tall, when in fact it is $i+1$ inches tall, has much better eyesight and a much greater ability to judge heights" than this agent (115).

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Taking the contrapositive, we have:

(1) $\forall i : K \neg h_i \to \neg h_{i+1}$

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Suppose that the agent reflects on the limitations of his visual discrimination and comes to know every instance of (1):

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Given these assumptions, it follows that for any j, if the agent knows that the height of the tree is not j inches, then he also knows that the height of the tree is not $j+1$ inches:

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Assuming $K\neg h_0$ holds, by repeating the steps of (3) - (6), we reach the conclusion $K\neg h_k$ by induction. Finally, by T, $K\neg h_k$ implies $\neg h_k$, contradicting our initial assumption of h_k.

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Formally, Williamson's observation is that for all $i, j \in \mathbb{N}$ with $j > i$:

$$\{K(K\neg h_i \to \neg h_{i+1}) \mid i \in \mathbb{N}\} \vdash_{\mathbf{K4}} K\neg h_i \to K\neg h_j.$$

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To model agents with limited discrimination, Williamson proposes epistemic models with non-transitive accessibility relations.

Non-transitive Models for Limited Discrimination

Suppose the agent has a fixed margin of error ϵ for judging the heights of the tree: so if the tree is height i, it is compatible with the agent's knowledge that its height is between $i - \epsilon$ and $i + \epsilon$.

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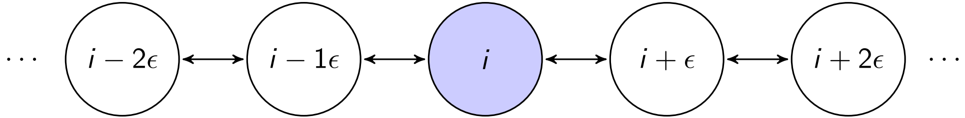
According to Williamson, part of the epistemic model for the agent should look like this (ignoring heights between i and $i \pm \epsilon$):



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$$\cdots \quad i - 2\epsilon \longleftrightarrow i - 1\epsilon \longleftrightarrow i \longleftrightarrow i + \epsilon \longleftrightarrow i + 2\epsilon \quad \cdots$$

Note: at the shaded world, $K\neg i + 2\epsilon \wedge \neg KK\neg i + 2\epsilon$ is true.

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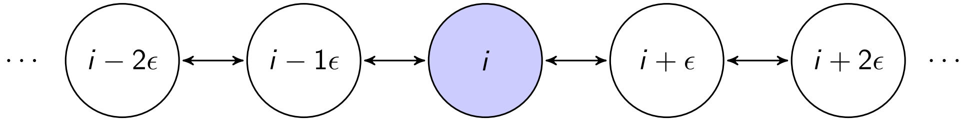

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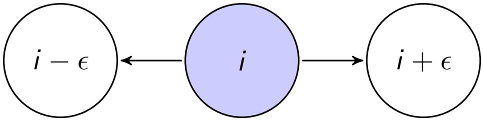
Compare the non-transitive model above with the transitive model:



Now $K\neg i+2\epsilon \wedge KK\neg i+2\epsilon$ is true at the shaded world.

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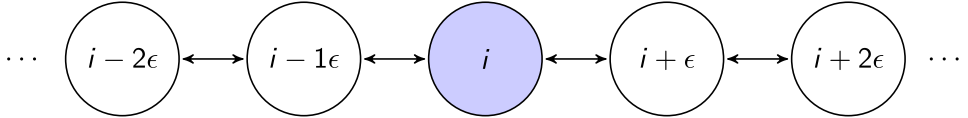

Note: at the shaded world, $K^l \neg 0$ (for some $l \in \mathbb{N}$) is *false*.

M. Gómez-Torrente. 1997.
"Two Problems for an Epistemicist View of Vagueness," *Philosophical Issues*.

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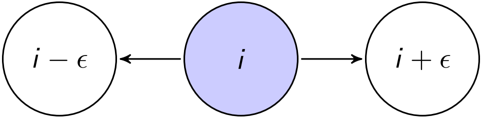

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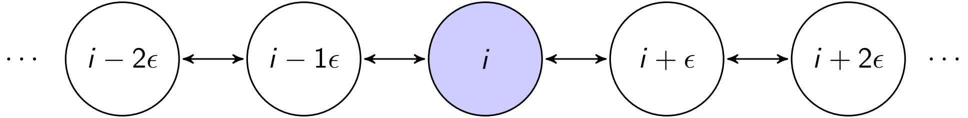

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What is preventing the agent from knowing that he knows that he knows ... (l times) ... that the tree is not 0 inches?

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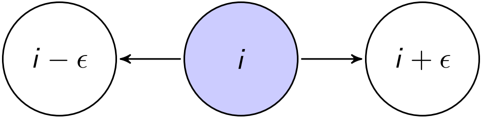

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Summary

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(Multi-agent) **KD45** is a logic of "belief"

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Two issues:

- Modeling awareness/unawareness
- Logics with both knowledge and belief operators