Reasoning about Knowledge and Beliefs

Lecture 3

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September 11, 2013

The Surprise Exam Paradox

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He concludes that the teacher cannot give him a surprise exam. But then he is surprised to receive an exam on, say, day $n-1$.

QUESTION: what went wrong in the student's reasoning?

The Designated Student Paradox

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A teacher shows her class of $n \geq 2$ clever logicians one gold star and $n-1$ silver stars. After lining the students up, single file, she walks behind each student and sticks one of the stars on his back. No student can see his own back, but each can see the backs of all students in front of him. The teacher announces that the student with the gold star will be **surprised** to learn that he has it.

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(This is clearly analogous to the surprise exam setup, but we have added a subtle but important difference. Think about it ...)

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But then the students pull the stars off their backs and it is, say, student $n-1$ who has the gold star, and he is surprised.

QUESTION: what went wrong in the student's reasoning?

Step 1: Choosing the Formalism (language)

To formalize the paradoxes, we use the epistemic language

$$\varphi ::= p_i \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi$$

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$K_i\varphi$ as "the student knows on the *morning* of day i that φ";

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For the designated student paradox, we read

$K_i\varphi$ as "the i-th student in line knows that φ";

p_i as "there is a gold star on the back of the i-th student".

Step 1: Choosing the Formalism (reasoning system)

To formalize the *reasoning* in the paradoxes, we will use the minimal "normal" modal proof system **K**, extending propositional logic with the following rule for each $i \in \mathbb{N}$ (Chellas 1980, §4.1):

$$\mathsf{RK}_i \ \frac{(\varphi_1 \wedge \cdots \wedge \varphi_m) \rightarrow \psi}{(K_i\varphi_1 \wedge \cdots \wedge K_i\varphi_m) \rightarrow K_i\psi},$$

which states that if the premise is a theorem, so is the conclusion.

Intuitively, RK_i says that the student on day i (or the i-th student) knows all the logical consequences of what he knows.

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For the surprise exam, (A) states that the student knows on the morning of day 1 that the teacher's announcement is true. (B) states that the student knows on the morning of day 1 that if the exam is on the afternoon of day 2, then the student will know on the morning of day 2 that it was not on day 1 (on the basis of memory).

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For the surprise exam, (A) states that the student knows on the morning of day 1 that the teacher's announcement is true. (B) states that the student knows on the morning of day 1 that if the exam is on the afternoon of day 2, then the student will know on the morning of day 2 that it was not on day 1 (on the basis of memory). Finally, (C) states that the student knows on the morning of day 1 that she will know on the morning of day 2 the part of the teacher's announcement about an *exam*.

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For the designated student, (A) states that student 1 knows that the teacher's announcement is true. (B) states that student 1 knows that if student 2 has the gold star, then student 2 knows that student 1 does not have the gold star (on the basis of seeing the silver star on student 1's back).

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Let us first show: $\{(A), (B), (C)\} \vdash_{\mathbf{K}} K_1(p_1 \wedge \neg K_1 p_1)$

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(1.2) $(K_2(p_1 \vee p_2) \wedge K_2 \neg p_1) \rightarrow K_2 p_2$ from (1.1) by RK_2

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(C) $K_1 K_2(p_1 \vee p_2)$ premise

(1) $(K_2(p_1 \vee p_2) \wedge K_2 \neg p_1) \rightarrow K_2 p_2$ using PL and RK_2

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(1) $(K_2(p_1 \vee p_2) \wedge K_2 \neg p_1) \to K_2 p_2$ using PL and RK_2

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(1) $(K_2(p_1 \vee p_2) \wedge K_2 \neg p_1) \rightarrow K_2 p_2$ using PL and RK_2

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(3) $K_1(K_2 \neg p_1 \rightarrow K_2 p_2)$ from (C) and (2) using PL and RK_1

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(C) $K_1 K_2(p_1 \lor p_2)$ premise

(1) $(K_2(p_1 \lor p_2) \land K_2 \neg p_1) \to K_2 p_2$ using PL and RK_2

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(3) $K_1(K_2 \neg p_1 \to K_2 p_2)$ from (C) and (2) using PL and RK_1

(4) $K_1 \neg(p_2 \land \neg K_2 p_2)$ from (B) and (3) using PL and RK_1

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(1) $(K_2(p_1 \lor p_2) \land K_2 \neg p_1) \to K_2 p_2$ using PL and RK_2

(2) $K_1((K_2(p_1 \lor p_2) \land K_2 \neg p_1) \to K_2 p_2)$ from (1) by Nec_1

(3) $K_1(K_2 \neg p_1 \to K_2 p_2)$ from (C) and (2) using PL and RK_1

(4) $K_1 \neg(p_2 \land \neg K_2 p_2)$ from (B) and (3) using PL and RK_1

(5) $K_1(p_1 \land \neg K_1 p_1)$ from (A) and (4) using PL and RK_1

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Given $\{(A), (B), (C)\} \vdash_{\mathbf{K}} K_1(p_1 \wedge \neg K_1 p_1)$, although we haven't yet derived a contradiction, we have derived something paradoxical.

Step 3: Showing Inconsistency with a Proof ($n = 2$)

Given $\{(A), (B), (C)\} \vdash_{\mathbf{K}} K_1(p_1 \wedge \neg K_1 p_1)$, although we haven't yet derived a contradiction, we have derived something paradoxical.

If we just add the "factivity" axiom T_1, $K_1\varphi \to \varphi$, or the "weak factivity" axiom J_1, $K_1\neg K_1\varphi \to \neg K_1\varphi$ (e.g., reading K as belief instead of knowledge), then we can derive a contradiction:

$$\{(A), (B), (C)\} \vdash_{\mathbf{KT_1}} \bot \text{ and } \{(A), (B), (C)\} \vdash_{\mathbf{KJ_1}} \bot.$$

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$$\{(A), (B), (C)\} \vdash_{\mathbf{KT_1}} \bot \text{ and } \{(A), (B), (C)\} \vdash_{\mathbf{KJ_1}} \bot.$$

Thus, we must reject either (A), (B), (C), or the rule RK_i...

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Comparison with $n = 3$ Case

The generalizations of (A), (B), and (C) to the $n = 3$ case are:

(A^3) $K_1((p_1 \wedge \neg K_1 p_1) \vee (p_2 \wedge \neg K_2 p_2) \vee (p_3 \wedge \neg K_3 p_3))$;

(B^3) $K_1(((p_2 \vee p_3) \rightarrow K_2 \neg p_1) \wedge (p_3 \rightarrow K_3 \neg(p_1 \vee p_2))$;

(C^3) $K_1(K_2(p_1 \vee p_2 \vee p_3) \wedge K_3(p_1 \vee p_2 \vee p_3))$.

Interestingly, as we will show later, these assumptions are *consistent* even if we make strong assumptions about knowledge.

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(C^3) $K_1(K_2(p_1 \vee p_2 \vee p_3) \wedge K_3(p_1 \vee p_2 \vee p_3))$.

If you think about the clever student's reasoning, he assumes that if he knows something, then he will continue to know it (or, for the designated student, then the students behind him in line know it):

$$4_1^< \quad K_1\varphi \rightarrow K_1 K_i \varphi \quad i > 1$$

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Using the axiom

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Using the axiom

$$4_1^< \quad K_1\varphi \to K_1 K_i \varphi \quad i > 1,$$

we can get into trouble starting from (A^3) and (B^3).
Indeed, the following result holds for any $n > 2$. See

Wes Holliday. "Simplifying the Surprise Exam." (email for manuscript)

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The generalizations of (A), (B), and (C) to the $n = 3$ case are:

(A^3) $K_1((p_1 \land \neg K_1 p_1) \lor (p_2 \land \neg K_2 p_2) \lor (p_3 \land \neg K_3 p_3))$;

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(C^3) $K_1(K_2(p_1 \lor p_2 \lor p_3) \land K_3(p_1 \lor p_2 \lor p_3))$.

For convenience, let's use the following abbreviation for "surprise":

$$S_i := (p_i \land \neg K_i p_i).$$

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The generalizations of (A), (B), and (C) to the $n = 3$ case are:

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(C^3) $K_1(K_2(p_1 \vee p_2 \vee p_3) \wedge K_3(p_1 \vee p_2 \vee p_3))$.

For convenience, let's use the following abbreviation for "surprise":

$$S_i := (p_i \wedge \neg K_i p_i).$$

Let us now show: $\{(A^3), (B^3)\} \vdash_{\mathbf{K4}_1^{<}} K_1(p \wedge \neg K_1 p_1)$

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(D^3) $K_1(K_2(S_1 \vee S_2 \vee S_3) \wedge K_3(p_1 \vee p_2 \vee p_3))$ from (A^3), $4_1^<$, RK_3, PL

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$(3,1)$ $(K_3(p_1 \vee p_2 \vee p_3) \wedge K_3\neg(p_1 \vee p_2)) \rightarrow K_3 p_3$ by PL and RK_3

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$(3,2)$ $K_1((K_3(p_1 \vee p_2 \vee p_3) \wedge K_3\neg(p_1 \vee p_2)) \to K_3 p_3)$ from $(3,1)$ by Nec_1

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$(3,3)$ $K_1(K_3\neg(p_1 \vee p_2) \rightarrow K_3 p_3)$ from (D^3), $(3,2)$ using RK_1 and PL

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$(3,1)$ $(K_3(p_1 \vee p_2 \vee p_3) \wedge K_3\neg(p_1 \vee p_2)) \to K_3 p_3$ by PL and RK_3

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$(3,3)$ $K_1(K_3\neg(p_1 \vee p_2) \to K_3 p_3)$ from (D^3), $(3,2)$ using RK_1 and PL

$(3,4)$ $K_1\neg S_3$ from (B^3), $(3,3)$ using RK_1 and PL

$(2,0)$ $K_1 K_2 \neg S_3$ from $(3,4)$ by $4_1^<$

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$(3,4)$ $K_1\neg S_3$ from (B^3), $(3,3)$ using RK_1 and PL

$(2,0)$ $K_1 K_2\neg S_3$ from $(3,4)$ by $4_1^<$

$(2,1)$ $(K_2(S_1 \vee S_2 \vee S_3) \wedge K_2\neg p_1 \wedge K_2\neg S_3) \rightarrow K_2 p_2$ by PL and RK_2

Let us now show: $\{(A^3),(B^3)\} \vdash_{\mathbf{K4}_1^<} K_1(p \wedge \neg K_1 p_1)$

(A^3) $K_1(S_1 \vee S_2 \vee S_3)$;

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$(3,2)$ $K_1((K_3(p_1 \vee p_2 \vee p_3) \wedge K_3\neg(p_1 \vee p_2)) \rightarrow K_3 p_3)$ from $(3,1)$ by Nec_1

$(3,3)$ $K_1(K_3\neg(p_1 \vee p_2) \rightarrow K_3 p_3)$ from (D^3), $(3,2)$ using RK_1 and PL

$(3,4)$ $K_1\neg S_3$ from (B^3), $(3,3)$ using RK_1 and PL

$(2,0)$ $K_1 K_2 \neg S_3$ from $(3,4)$ by $4_1^<$

$(2,1)$ $(K_2(S_1 \vee S_2 \vee S_3) \wedge K_2\neg p_1 \wedge K_2\neg S_3) \rightarrow K_2 p_2$ by PL and RK_2

$(2,2)$ $K_1((K_2(S_1 \vee S_2 \vee S_3) \wedge K_2\neg p_1 \wedge K_2\neg S_3) \rightarrow K_2 p_2)$ from $(2,1)$ by Nec_1

Let us now show: $\{(A^3), (B^3)\} \vdash_{\mathbf{K4}_1^<} K_1(p \wedge \neg K_1 p_1)$

(A^3) $K_1(S_1 \vee S_2 \vee S_3)$;

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$(3,3)$ $K_1(K_3 \neg(p_1 \vee p_2) \to K_3 p_3)$ from (D^3), $(3,2)$ using RK_1 and PL

$(3,4)$ $K_1 \neg S_3$ from (B^3), $(3,3)$ using RK_1 and PL

$(2,0)$ $K_1 K_2 \neg S_3$ from $(3,4)$ by $4_1^<$

$(2,1)$ $(K_2(S_1 \vee S_2 \vee S_3) \wedge K_2 \neg p_1 \wedge K_2 \neg S_3) \to K_2 p_2$ by PL and RK_2

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$(3,4)$ $K_1\neg S_3$ from (B^3), $(3,3)$ using RK_1 and PL

$(2,0)$ $K_1 K_2\neg S_3$ from $(3,4)$ by $4_1^<$

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$(2,0)$ $K_1 K_2\neg S_3$ from $(3,4)$ by $4_1^<$

$(2,1)$ $(K_2(S_1 \vee S_2 \vee S_3) \wedge K_2\neg p_1 \wedge K_2\neg S_3) \rightarrow K_2 p_2$ by PL and RK_2

$(2,2)$ $K_1((K_2(S_1 \vee S_2 \vee S_3) \wedge K_2\neg p_1 \wedge K_2\neg S_3) \rightarrow K_2 p_2)$ from $(2,1)$ by Nec_1

$(2,3)$ $K_1(K_2\neg p_1 \rightarrow K_2 p_2)$ from (D^3), $(2,0)$, $(2,2)$ using RK_1 and PL

$(2,4)$ $K_1\neg S_2$ from (B^3), $(2,3)$ using RK_1 and PL

$(2,5)$ $K_1 S_1$ from (A^3), $(3,4)$, $(2,4)$ using RK_1 and PL

Comparison with $n = 3$ Case

(A^3) $K_1((p_1 \wedge \neg K_1 p_1) \vee (p_2 \wedge \neg K_2 p_2) \vee (p_3 \wedge \neg K_3 p_3))$;

(B^3) $K_1(((p_2 \vee p_3) \to K_2 \neg p_1) \wedge (p_3 \to K_3 \neg (p_1 \vee p_2)))$.

As before, given $\{(A^3), (B^3)\} \vdash_{\mathbf{K4}_1^<} K_1(p \wedge \neg K_1 p_1)$, we also have:

$$\{(A^3), (B^3)\} \vdash_{\mathbf{KT_1 4}_1^<} \bot \text{ and } \{(A^3), (B^3)\} \vdash_{\mathbf{KJ_1 4}_1^<} \bot.$$

Comparison with $n = 3$ Case

(A^3) $K_1((p_1 \wedge \neg K_1 p_1) \vee (p_2 \wedge \neg K_2 p_2) \vee (p_3 \wedge \neg K_3 p_3))$;

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As before, given $\{(A^3), (B^3)\} \vdash_{\mathbf{K4}_1^<} K_1(p \wedge \neg K_1 p_1)$, we also have:

$$\{(A^3), (B^3)\} \vdash_{\mathbf{KT_14}_1^<} \bot \text{ and } \{(A^3), (B^3)\} \vdash_{\mathbf{KJ_14}_1^<} \bot.$$

Thus, we must reject (A^3), (B^3), the rule RK or the axiom

$$4_1^< \quad K_1\varphi \to K_1 K_i \varphi \quad i > 1.$$

Step 4: Showing Consistency with a Model

(A^3) $K_1((p_1 \wedge \neg K_1 p_1) \vee (p_2 \wedge \neg K_2 p_2) \vee (p_3 \wedge \neg K_3 p_3))$;

(B^3) $K_1(((p_2 \vee p_3) \rightarrow K_2 \neg p_1) \wedge (p_3 \rightarrow K_3 \neg(p_1 \vee p_2))$;

(C^3) $K_1(K_2(p_1 \vee p_2 \vee p_3) \wedge K_3(p_1 \vee p_2 \vee p_3))$.

Let's now establish the previous claim about the consistency of (A^3), (B^3), (C^3), even with strong assumptions about knowledge.

Step 4: Showing Consistency with a Model

(A^3) $K_1((p_1 \wedge \neg K_1 p_1) \vee (p_2 \wedge \neg K_2 p_2) \vee (p_3 \wedge \neg K_3 p_3));$

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Even adding to **K** the T schema, $K_i\varphi \to \varphi$, and the 5 schema, $\neg K_i\varphi \to K_i \neg K_i \varphi$, to obtain the strong system **S5**, we have:

$$\{(A^3), (B^3), (C^3)\} \nvdash_{\mathbf{S5}} \bot.$$

Step 4: Showing Consistency with a Model

(A^3) $K_1((p_1 \wedge \neg K_1 p_1) \vee (p_2 \wedge \neg K_2 p_2) \vee (p_3 \wedge \neg K_3 p_3))$;

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$$\{(A^3), (B^3), (C^3)\} \nvdash_{\mathbf{S5}} \bot.$$

To show this, we'll turn to models for the epistemic language.

Step 4: Showing Consistency with a Model

The logic **S5** is sound with respect to the class of relational models $\mathcal{M} = \langle W, \{R_i\}_{i \in \mathbb{N}}, V \rangle$ where each R_i is an *equivalence relation*, i.e., reflexive, symmetric, and transitive.

Step 4: Showing Consistency with a Model

The logic **S5** is sound with respect to the class of relational models $\mathcal{M} = \langle W, \{R_i\}_{i \in \mathbb{N}}, V \rangle$ where each R_i is an *equivalence relation*, i.e., reflexive, symmetric, and transitive.

Thus, if we can construct such a model in which (A^3), (B^3), and (C^3) are all true, then we have $\{(A^3), (B^3), (C^3)\} \nvdash_{\mathbf{S5}} \perp$.

Step 4: Showing Consistency with a Model

(A^3) $K_1((p_1 \wedge \neg K_1 p_1) \vee (p_2 \wedge \neg K_2 p_2) \vee (p_3 \wedge \neg K_3 p_3))$;

(B^3) $K_1(((p_2 \vee p_3) \to K_2 \neg p_1) \wedge (p_3 \to K_3 \neg (p_1 \vee p_2))$;

(C^3) $K_1(K_2(p_1 \vee p_2 \vee p_3) \wedge K_3(p_1 \vee p_2 \vee p_3))$.

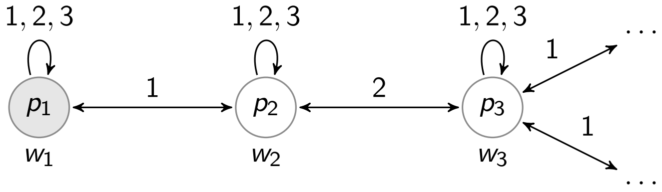

Figure: model establishing **S5**-consistency of $(A^3), (B^3), (C^3)$.

Observe that $K_1 \neg p_3 \to K_1 K_2 \neg p_3$, an instance of $4_1^<$, is false at w_1.

Summary of What We've Seen

Here's a summary of what we've seen:

- $\{(A^2), (B^2), (C^2)\} \vdash_{\mathbf{K}} K_1(p_1 \wedge \neg K_1)$;
- $\{(A^2), (B^2), (C^2)\} \vdash_{\mathbf{KJ_1}} \bot$ and $\{(A^2), (B^2), (C^2)\} \vdash_{\mathbf{KT_1}} \bot$;

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- $\{(A^3),(B^3),(C^3)\} \nvdash_{\mathbf{S5}} \bot$.

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- $\{(A^2), (B^2), (C^2)\} \vdash_{\mathbf{K}} K_1(p_1 \wedge \neg K_1)$;
- $\{(A^2), (B^2), (C^2)\} \vdash_{\mathbf{KJ_1}} \perp$ and $\{(A^2), (B^2), (C^2)\} \vdash_{\mathbf{KT_1}} \perp$;
- $\{(A^3), (B^3), (C^3)\} \nvdash_{\mathbf{S5}} \perp$.
- $\{(A^3), (B^3)\} \vdash_{\mathbf{K4_1^<}} K_1(p_1 \wedge \neg K_1)$;
- $\{(A^3), (B^3)\} \vdash_{\mathbf{KJ_14_1^<}} \perp$ and $\{(A^3), (B^3)\} \vdash_{\mathbf{KT_14_1^<}} \perp$;

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- $\{(A^2), (B^2), (C^2)\} \vdash_{\mathbf{KJ_1}} \bot$ and $\{(A^2), (B^2), (C^2)\} \vdash_{\mathbf{KT_1}} \bot$;
- $\{(A^3), (B^3), (C^3)\} \nvdash_{\mathbf{S5}} \bot$.
- $\{(A^3), (B^3)\} \vdash_{\mathbf{K4_1^<}} K_1(p_1 \wedge \neg K_1)$;
- $\{(A^3), (B^3)\} \vdash_{\mathbf{KJ_14_1^<}} \bot$ and $\{(A^3), (B^3)\} \vdash_{\mathbf{KT_14_1^<}} \bot$;

With these facts, one can make a strong case that the culprit behind the paradoxes is the (mistaken) $4_1^<$ axiom, $K_1\varphi \to K_1K_i\varphi$ $(i > 1)$. But we don't have time to explain this solution. See

Wes Holliday. *"Simplifying the Surprise Exam."*. (email for manuscript).