Introductory Notes on Modal Logic

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These short notes are intended to introduce some of the basic concepts of *Modal Logic*. The primary goal is to provide students in Philosophy 370 at the University of Maryland, College Park with a study guide that will complement the lectures on modal logic. There are many textbooks that you can consult for more information. The following is a list of some texts (this is not a complete list, but a pointer to books that I have found particularly useful).

- *Modal Logic for Open Minds* by Johan van Benthem. A new textbook on modal logic providing a modern introduction to modal logic.
- *Modal Logic for Philosophers* by James Garson. An introduction to modal logic geared towards Philosophy students.
- *Modal Logic* by Brian Chellas. A nice introduction to modal logic though somewhat outdated.
- The *Modal Logic* entry at the Stanford Encyclopedia of Philosophy (`http://plato.stanford.edu/entries/logic-modal/`). This entry was written by James Garson and provides a nice overview of the philosophical applications of modal logic.

There are also more advanced books that you should keep on your radar.

- *Handbook of Modal Logic* edited by Johan van Benthem, Patrick Blackburn and Frank Wolter. This very extensive volume represents the current state-of-affairs in modal logic.
- *Modal Logic* by Patrick Blackburn, Maarten de Rijke and Yde Venema. An advanced, but very accessible, textbook focusing on the main technical results in the area.
- *First Order Modal Logic* by Melvin Fitting and Elliot Mehdelsohn. The focus here is on *first-order* modal logic (as opposed to propositional modal logic which is the focus of most of the other texts mentioned here). This text provides both a philosophical and technical introduction to this fascinating area.

1 Syntax and Semantics of Modal Logic

What is a modal? A *modal* is anything that qualifies the truth of a sentence. There are many ways to qualify the truth of a statement in natural language. For example, each of the phrases below can be used to complete the sentence: John _______ happy.

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- is necessarily
- is possibly
- is known/believed (by Ann) to be
- is probably
- is permitted to be
- is obliged to be
- is now
- will be
- can do something to ensure that he is

The basic modal language is a generic formal language that has been used to reason about situations involving modal notions. This language is defined as follows:

Definition 1.1 (The Basic Modal Language) Let $\mathcal{S} = \{p, q, r, \ldots\}$ be a set of sentence letters, or atomic propositions. We also include two special propositions '$\top$' and '$\bot$' meaning 'true' and 'false' respectively. The set of well-formed formulas of modal logic is the smallest set generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \Diamond\varphi$$

where $p \in \mathcal{S}$. Let $\mathcal{L}$ denote the basic modal language. $\triangleleft$

Examples of modal formulas include: $\Box\bot$, $\Box\Diamond\top$, $p \to \Box(q \wedge r)$, and $\Box(p \to (q \vee \Diamond r) \leftrightarrow \Diamond\Box p)$.

One language, many readings. There are many possible readings for the modal operators '$\Box$' and '$\Diamond$'. Here are some samples:

- **Alethic Reading**: $\Box\varphi$ means 'φ is necessary' and $\Diamond\varphi$ means 'φ is possible'.
- **Deontic Reading**: $\Box\varphi$ means 'φ is obligatory' and $\Diamond\varphi$ means 'φ is permitted'. In this literature, typically 'O' is used instead of '$\Box$' and 'P' instead of '$\Diamond$'.
- **Epistemic Reading**: $\Box\varphi$ means 'φ is known' and $\Diamond\varphi$ means 'φ is consistent with the current information'. In this literature, typically 'K' is used instead of '$\Box$' and 'L' instead of '$\Diamond$'.
- **Doxastic Reading**: $\Box\varphi$ means 'φ is believed' and $\Diamond\varphi$ means 'φ is (doxastically) possible'. In this literature, typically 'B' is used instead of '$\Box$'.
- **Temporal Reading**: $\Box\varphi$ means 'φ will always be true' and $\Diamond\varphi$ means 'φ will be true at some point in the future'.
- **Provability Reading**: $\Box\varphi$ means 'there is a proof of φ (for example, in Peano Arithmetic)' and $\Diamond\varphi$ means 'φ is consistent (given a proof system such as Peano Arithmetic)'.

There are many interesting arguments involving modal notions. Below I give two examples, both of which have been widely discussed by philosophers.

Example 1.2 (Aristotle's Sea Battle Argument) A general is contemplating whether or not to give an order to attack. The general reasons as follows:

1. If I give the order to attack, then, necessarily, there will be a sea battle tomorrow
2. If not, then, necessarily, there will not be one.
3. Now, I give the order or I do not.
4. Hence, either it is necessary that there is a sea battle tomorrow or it is necessary that none occurs.

The conclusion is that either it is inevitable that there is a sea battle tomorrow or it is inevitable that there is no battle. So, why should the general bother giving the order? There are two possible formalizations of this argument corresponding to different readings of "if A then necessarily B":

$$\begin{array}{l} A \to \Box B \\ \neg A \to \Box\neg B \\ A \vee \neg A \\ \hline \Box B \vee \Box\neg B \end{array} \qquad\qquad \begin{array}{l} \Box(A \to B) \\ \Box(\neg A \to \neg B) \\ A \vee \neg A \\ \hline \Box B \vee \Box\neg B \end{array}$$

Are these two formalizations the same? If not, which argument is valid? ◁

The second example, provided by J. Forrester in 1984, involves the Deontic reading of modal logic.

Example 1.3 (The Gentle Murder Paradox) Suppose that Jones murders Smith. Accepting the principle that 'If Jones murders Smith, then Jones ought to murder Smith gently', we can argue that, in fact, Jones *ought* to murder Smith as follows:

1. Jones murders Smith. (M)
2. If Jones murders Smith, then Jones ought to murder Smith gently. ($M \to OG$)
3. Jones ought to murder Smith gently. (OG)
4. If Jones murders Smith gently, then Jones murders Smith. ($G \to M$)
5. If Jones ought to murder Smith gently, then Jones ought to murder Smith. ($OG \to OM$)
6. Jones ought to murder Smith. (OM)

Is this argument valid? Note that reasoning from statement 4. to statement 5. follows a general modal reasoning pattern: if '$X \to Y$' has been established, then we can establish '$\Box X \to \Box Y$'. ◁

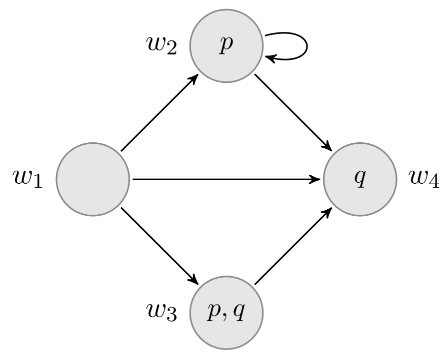
In order to answer the questions in the examples above, we need a natural semantics for the basic modal language.

Question 1.4 *Can we give a truth-table semantics for the basic modal language? (***Hint***: there are only 4 possible truth-table for a unary operator. Suppose we want $\Box A \to A$ to be valid (i.e., true regardless of the truth value assigned to A), but allow $A \to \Box A$ and $\neg\Box A$ to be false (i.e., for each formula, there is a possible assignment of truth values to A which makes the formulas false).*

A semantics for the basic modal language was developed by Saul Kripke, Stag Kanger, Jaakko Hinitkka and others in the 1960s and 1970s. Formulas are interpreted over graph-like structures:

Definition 1.5 (Relational Structure) A **Relational Structure** (also called a possible worlds model, Kripke model or a modal model) is a triple $\mathcal{M} = \langle W, R, V \rangle$ where W is a nonempty set (elements of W are called **states**), R is a relation on W (formally, $R \subseteq W \times W$) and V is a **valuation function** assigning truth values $V(p, w)$ to atomic propositions p at state w (formally $V : \mathcal{S} \times W \to \{0, 1\}$ where $\mathcal{S}$ is the set of sentence letters). ◁

Example 1.6 Often relational structures are drawn instead of formally defined. For example, the following picture represents the relational structure $\mathcal{M} = \langle W, R, V \rangle$ where $W = \{w_1, w_2, w_3, w_4\}$, $R = \{(w_1, w_2), (w_1, w_3), (w_1, w_4), (w_2, w_2), (w_2, w_4), (w_3, w_4)\}$ and $V(p, w_2) = V(p, w_3) = V(q, w_3) = V(q, w_4) = 1$ (with all other propositional variables assigned 0 at the states).



◁

Formulas of the basic modal language are interpreted at states in a relational structure.

Definition 1.7 (Truth of Modal Formulas) Truth of a modal formula φ at a state w in a relational structure $\mathcal{M} = \langle W, R, V \rangle$, denoted $\mathcal{M}, w \models \varphi$ is defined inductively as follows:

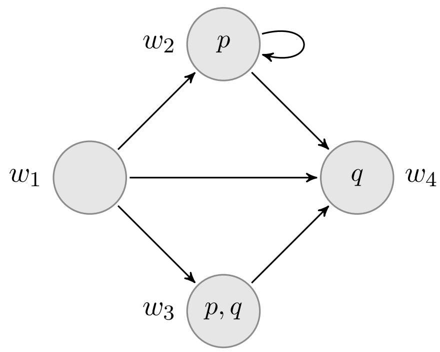
1. $\mathcal{M}, w \models p$ iff $V(p, w) = 1$ (where $p \in \mathcal{S}$)
2. $\mathcal{M}, w \models \top$ and $\mathcal{M}, w \not\models \bot$
3. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
4. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
5. $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
6. $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$ ◁

Two remarks about this definition. First, note that truth for the other boolean connectives ($\to, \vee, \leftrightarrow$) is not given in the above definition. This is not necessary since these connectives are *definable* from '$\neg$' and '$\wedge$'. [1] As an exercise, make sure you can specify the truth definition in the style of the Definition above for each of the boolean connectives not mentioned. Second, note the analogy between '$\Box$' and a universal quantifier and '$\Diamond$' and a existential quantifier.

[1] For example, $\varphi \to \psi$ can be defined as (i.e., is logically equivalent to) $\neg(\varphi \wedge \neg\psi)$.

Question 1.8 *Let $\mathcal{M} = \langle W, R, V\rangle$ be a relational model. Give the recursive definition of a function $\overline{V} : \mathcal{L} \to \wp(W)$ so that $\overline{V}(\varphi) = \{w \in W \mid \mathcal{M}, w \models \varphi\}$ (recall that $\wp(W) = \{X \mid X \subseteq W\}$ is the powerset of W).*

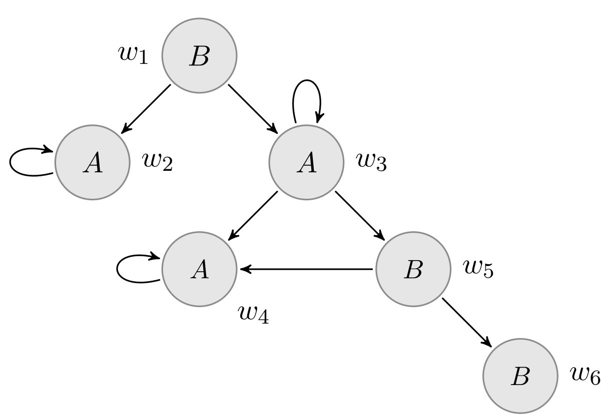
Example 1.9 To illustrate the above definition of truth of modal formula, recall the relational structure from Example 1.6:



- $\mathcal{M}, w_3, \models \Box q$: w_4 is the only worlds accessible from w_3 and q is true at w_4.
- $\mathcal{M}, w_1 \models \Diamond q$: there is a state accessible from w_1 (namely w_3) where q is true.
- $\mathcal{M}, w_1 \models \Diamond\Box q$: w_3 is accessible from w_1 and q is true in all of the worlds accessible from w_3.
- $\mathcal{M}, w_4 \models \Box\bot$: there are no worlds accessible from w_4, so any formula beginning with '$\Box$' will be true (this is analogous to the fact the universal sentences are true in any first-order structure where the domain is empty). Similarly, any formula beginning with a '$\Diamond$' will be false (again, this is analogous to the fact that existential statements are false in first-order structures with empty domains). $\triangleleft$

For an extended discussion surrounding the interpreting modal formulas in relational structures, see Chapter 2 of *Modal Logic for Open Minds* by Johan van Benthem.

Question 1.10 *Consider the following relational structure.*



1. $\Box A \to \Box\Box A$
2. $\Box\Box A \to \Box A$
3. $\Diamond(\Diamond A \wedge \Diamond B)$
4. $\Diamond\Box\bot$
5. $\Box(\Box A \to A) \to \Box A$

For each formula to the right, list the states where the formula is true.

2 Modal Validity

Definition 2.1 (Modal Validity) A modal formula φ is **valid in a relational structure** $\mathcal{M} = \langle W, R, V\rangle$, denoted $\mathcal{M} \models \varphi$, provided $\mathcal{M}, w \models \varphi$ for each $w \in W$. A modal formula φ is **valid**, denoted $\models \varphi$, provided φ is valid in all relational structures. $\lhd$

In order to show that a modal formula φ is valid, it is enough to argue informally that φ is true at an arbitrary state in an arbitrary relational structure. On the other hand, to show a modal formula φ is not valid, one must provide a counter example (i.e., a relational structure and state where φ is false).

Fact 2.2 $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ *is valid.*

Proof. Suppose $\mathcal{M} = \langle W, R, V\rangle$ is an arbitrary relational structure and $w \in W$ an arbitrary state. We will show that $\mathcal{M}, w \models \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$. We first show that if $\mathcal{M}, w \models \Box\varphi$ then $\mathcal{M}, w \models \neg\Diamond\neg\varphi$. If $\mathcal{M}, w \models \Box\varphi$ then for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$. Suppose (to get a contradiction) that $\mathcal{M}, w \models \Diamond\neg\varphi$. Then there is some v' such that wRv' and $\mathcal{M}, v' \models \neg\varphi$. Therefore, since wRv' we have $\mathcal{M}, v' \models \varphi$ and $\mathcal{M}, v' \models \neg\varphi$ which means $\mathcal{M}, v' \not\models \varphi$. But this is a contradiction, so $\mathcal{M}, w \not\models \Diamond\neg\varphi$. Hence, $\mathcal{M}, w \models \neg\Diamond\neg\varphi$.

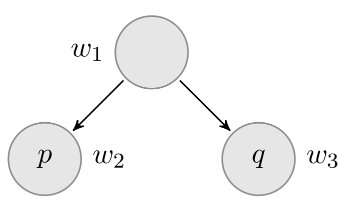
We now show that if $\mathcal{M}, w \models \neg\Diamond\neg\varphi$ then $\mathcal{M}, w \models \Box\varphi$. Suppose that $\mathcal{M}, w \models \neg\Diamond\neg\varphi$. Then there is no state v such that wRv and $\mathcal{M}, v \models \neg\varphi$. Let v be any element of W such that wRv. Then $\mathcal{M}, w \models \varphi$ (since otherwise there would be an accessible state satisfying $\neg\varphi$). Therefore, $\mathcal{M}, w \models \Box\varphi$. QED

Fact 2.3 $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ *is valid.*

Proof. Suppose $\mathcal{M} = \langle W, R, V\rangle$ is an arbitrary relational structure and $w \in W$ an arbitrary state. We will show $\mathcal{M}, w \models \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$. Suppose that $\mathcal{M}, w \models \Box\varphi \wedge \Box\psi$. Then $\mathcal{M}, w \models \Box\varphi$ and $\mathcal{M}, w \models \Box\psi$. Suppose that $v \in W$ and wRv. Then $\mathcal{M}, v \models \varphi$ and $\mathcal{M}, v \models \psi$. Hence, $\mathcal{M}, v \models \varphi \wedge \psi$. Since v is an arbitrary state accessible from w, we have $\mathcal{M}, w \models \Box(\varphi \wedge \psi)$. QED

Fact 2.4 $(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$ *is not valid.*

Proof. We must find a relational structure that has a state where $(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$ is false. Consider the following relational structure:

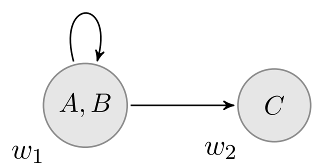


Call this relational structure $\mathcal{M}$. We have $\mathcal{M}, w_1 \models \Diamond p \wedge \Diamond q$ (why?), but $\mathcal{M}, w_1 \not\models \Diamond(p \wedge q)$ (why?). Hence, $\mathcal{M}, w_1 \not\models (\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$. QED

Question 2.5 *Determine which of the following formulas valid (prove your answers):*

1. $\Box\varphi \to \Diamond\varphi$
2. $\Box(\varphi \vee \neg\varphi)$
3. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
4. $\Box\varphi \to \varphi$
5. $\varphi \to \Box\Diamond\varphi$
6. $\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$

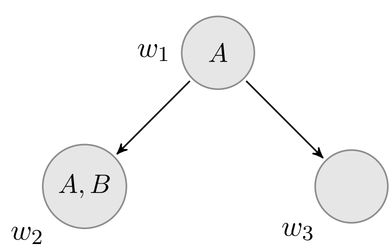
We can now see why the two formalizations of Aristotle's Sea Battle Argument (cf. Exercise 1.2) are not "equivalent". They would be the "same" if $\Box(A \to B)$ is (modally) equivalent to $A \to \Box B$. That is if $\Box(A \to B) \leftrightarrow (A \to \Box B)$ is valid. The following relational structure shows that this is not the case:



Here $\Box(A \to B)$ is true at w_1 but $A \to \Box B$ is not true at w_1 (why?). Furthermore, the second formalization of Aristotle's Sea Battle Argument is not valid:

$$\begin{array}{l} \Box(A \to B) \\ \Box(\neg A \to \neg B) \\ A \vee \neg A \\ \hline \Box B \vee \Box\neg B \end{array}$$

To show this, we must find a relational structure that has a state where all of the premises are true but the conclusion ($\Box B \vee \Box\neg B$) is false. The following relational structure does the trick (w_1 satisfies all of the premises but not the conclusion):

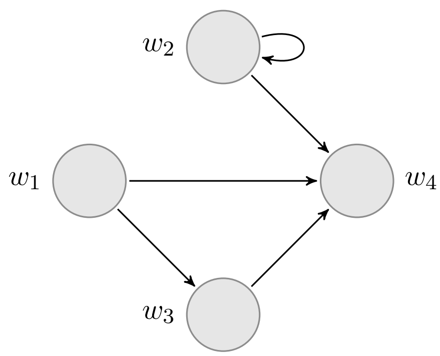


3 Definability

Question 1.8 shows that we can assign to every modal formula φ a set of states in a relational structure $\mathcal{M} = \langle W, R, V\rangle$ (i.e., the set $\overline{V}(\varphi)$ of states where φ is true in $\mathcal{M}$). We sometime write $(\varphi)^{\mathcal{M}}$ for this set. What about the converse: given and arbitrary set, when does a formula uniquely pick out that set?

Definition 3.1 (Definable Subsets) Let $\mathcal{M} = \langle W, R, V\rangle$ be a relational structure. A set $X \subseteq W$ is **definable in** $\mathcal{M}$ provided $X = (\varphi)^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \varphi\}$ for some modal formula φ. $\lhd$

Example 3.2 All four of the states in the relational structure below are uniquely defined by a modal formula:



- $\{w_4\}$ is defined by $\Box\bot$
  (w_4 *is the only "dead-end" state*)
- $\{w_3\}$ is defined by $\Diamond\Box\bot \wedge \Box\Box\bot$
  (w_3 *can only see a "dead-end" state*)
- $\{w_2\}$ is defined by $\Diamond\Diamond\Diamond\top$
  (w_2 *is the only state where 3 steps can be taken*)
- $\{w_1\}$ is defined by $\Diamond(\Diamond\Box\bot \wedge \Box\Box\bot)$
  (w_1 *is the only state that can see* w_3)

Given the above observations, it is not hard to see that *all* subsets of $W = \{w_1, w_2, w_3, w_4\}$ are definable (why?). However, note that even in finite relational structures, not all subsets may be definable. A problem can arise if states cannot be distinguished by modal formulas. For example, if the reflexive arrow is dropped in the relational structure above, then w_2 and w_3 *cannot* be distinguished by a modal formula (there are ways to formally prove this, but see if you can informally argue why w_2 and w_3 cannot be distinguished). $\lhd$

The next two definitions make precise what it means for two states to be *indistinguishable* by a modal formula.

Definition 3.3 (Modal Equivalence) Let $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ be two relational structures. We say $\mathcal{M}_1, w_2$ and $\mathcal{M}_2, w_2$ are **modally equivalent** provided

$$\text{for all modal formulas } \varphi,\ \mathcal{M}_1, w_1 \models \varphi \text{ iff } \mathcal{M}_2, w_2 \models \varphi$$

We write $\mathcal{M}_1, w_1 \leftrightsquigarrow \mathcal{M}_2, w_2$ if $\mathcal{M}_1, w_1$ and $\mathcal{M}_2, w_2$ are modally equivalent. (Note that it is assumed $w_1 \in W_1$ and $w_2 \in W_2$) $\lhd$

Definition 3.4 (Bisimulation) Let $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ be two relational structures. A nonempty relation $Z \subseteq W_1 \times W_2$ is called a **bisimulation** provided for all $w_1 \in W_1$ and $w_2 \in W_2$, if $w_1 Z w_2$ then

1. (atomic harmony) For all $p \in \mathcal{S}$, $V_1(w_1, p) = V_2(w_2, p)$.
2. (zig) If $w_1 R_1 v_1$ then there is a $v_2 \in W_2$ such that $w_2 R_2 v_2$ and $v_1 Z v_2$.
3. (zag) If $w_2 R_2 v_2$ then there is a $v_1 \in W_1$ such that $w_1 R_1 v_1$ and $v_1 Z v_2$.

We write $\mathcal{M}_1, w_1 \,\underline{\leftrightarrow}\, \mathcal{M}_2, w_2$ if there is a bisimulation relating w_1 with w_2. $\lhd$

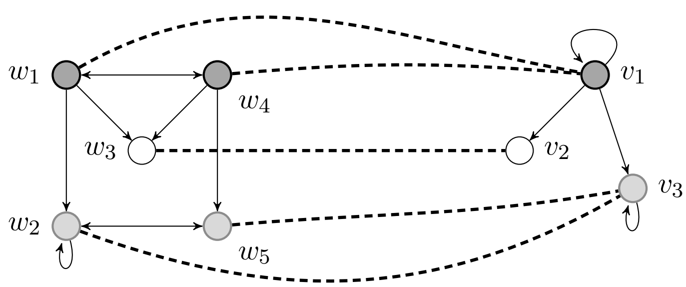
Definition 3.3 and 3.4 provide two concrete ways to answer the question: *when are two states the same*? The following questions are straightforward consequences of the relevant definitions.

Question 3.5 *1. Prove $\leftrightsquigarrow$ and $\underline{\leftrightarrow}$ are equivalence relations.*

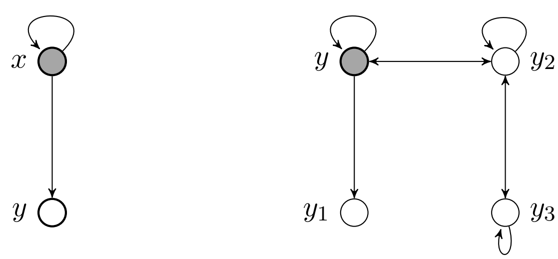
2. *Prove that if X is a definable subset of $\mathcal{M} = \langle W, R, V\rangle$, then X is closed under the $\leftrightsquigarrow$ relation (if $w \in X$ and $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}, v$ then $v \in X$).*

3. *Prove that there is a largest bisimulation: given $\{Z_i \mid i \in I\}$ a set of bisimulations relating the relational structures $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ (i.e., for each $i \in I$, $Z_i \subseteq W_1 \times W_2$ satisfies Definition 3.4), show the relation $Z = \bigcup_{i\in I} Z_i$ is a bisimulation.*

Example 3.6 The dashed lines is a bisimulation between the following two relational structures (for simplicity, we do assume that all atomic propositions are false):



On the other hand, there is no bisimulation relating the state x and y in the following two relational structures:



Using Lemma 3.7 below, we can *prove* that there is no bisimulation relating x and y. We first note that $\Box(\Diamond\Box\bot \vee \Box\bot)$ is true at state x but not true at state y. Then by Lemma 3.7, x and y cannot be bisimilar. $\triangleleft$

Lemma 3.7 (Modal Invariance Lemma) *Suppose $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ are relational structures. For all $w \in W_1$ and $v \in W_2$, if $\mathcal{M}_1, w \underline{\leftrightarrow} \mathcal{M}_2, v$ then $\mathcal{M}_1, w \leftrightsquigarrow \mathcal{M}_2, v$.*

Proof. Suppose that $\mathcal{M}_1, w \underline{\leftrightarrow} \mathcal{M}_2, v$. Then, there is a bisimulation Z such that wZv. The proof is by induction on the structure of φ. The base case is when φ is p, an atomic proposition. By the atomic harmony condition, since wZv, we have $V_1(w,p) = V_2(v,p)$. Hence, $\mathcal{M}_1, w \models p$ iff $\mathcal{M}_2, v \models p$.

There are three cases:

Case 1: φ is $\psi_1 \wedge \psi_2$. Then,

$$\begin{array}{rlll}\mathcal{M}_1, w \models \psi_1 \wedge \psi_2 & \text{iff} & \mathcal{M}_1, w \models \psi_1 \text{ and } \mathcal{M}_1, w \wedge \psi_2 & \text{(Def. of Truth)} \\ & \text{iff} & \mathcal{M}_2, v \models \psi_1 \text{ and } \mathcal{M}_2, v \models \psi_2 & \text{(Induction hypothesis)} \\ & \text{iff} & \mathcal{M}_2, v \models \psi_1 \wedge \psi_2 & \text{(Def. of truth)}\end{array}$$

Case 2: φ is $\neg\psi$. Then,

$$\begin{array}{rlll}\mathcal{M}_1, w \models \neg\psi & \text{iff} & \mathcal{M}_1, w \not\models \psi & \text{(Def. of Truth)} \\ & \text{iff} & \mathcal{M}_2, v \not\models \psi & \text{(Induction hypothesis)} \\ & \text{iff} & \mathcal{M}_2, v \models \neg\psi & \text{(Def. of truth)}\end{array}$$

Case 3: φ is $\Box\psi$. Suppose that $\mathcal{M}_1, w \models \Box\psi$. Then for each w', if wR_1w', then $\mathcal{M}_1, w' \models \psi$. We will show that $\mathcal{M}_2, v \models \Box\psi$. The v' be any state in W_2 with vR_2v'. By the zig condition, there is a $w' \in W_1$ such that wR_1w' and $w'Zv'$. Since $\mathcal{M}_1, w \models \Box\psi$ and wR_1w', we have $\mathcal{M}_1, w' \models \psi$. By the induction hypothesis, $\mathcal{M}_2, v' \models \psi$. Since v' is an arbitrary state with vR_2v', we have $\mathcal{M}_2, v \models \Box\psi$. The converse direction is similar (it makes use of the zag condition). QED

Lemma 3.8 *Suppose $\mathcal{M}_1 = \langle W_1, R_1, V_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2\rangle$ are* finite *relational structures. If $\mathcal{M}_1, w_1 \leftrightsquigarrow \mathcal{M}_2, w_2$ then $\mathcal{M}_1, w_1 \,\underline{\leftrightarrow}\, \mathcal{M}_2, w_2$.*

Proof. We show that $\leftrightsquigarrow$ is a bisimulation. The atomic harmony condition is obvious. We prove the zag condition. Suppose that $\mathcal{M}_1, w_1 \leftrightsquigarrow \mathcal{M}_2, w_2$, $w_2R_2v_2$, but there is no v_1 such that $w_1R_1v_1$ and $\mathcal{M}_1, v_1 \leftrightsquigarrow \mathcal{M}_2, v_2$. Note that there are only finitely many states that are accessible from w_1. That is, $\{w \mid w_1R_1w\}$ is a finite set. Suppose that $\{w \mid w_1R_1w\} = \{w^1, w^2, \ldots, w^m\}$. By assumption, for each w^i we have $\mathcal{M}_1, w^i \not\leftrightsquigarrow \mathcal{M}_2, v_2$. Hence, for each w^i, there is a formula φ_i such that $\mathcal{M}_1, w^i \not\models \varphi_i$ but $\mathcal{M}_2, v_2 \models \varphi_i$. Then, $\mathcal{M}_2, v_2 \models \bigwedge_{i=1,\ldots,m} \varphi_i$. Since $w_2R_2v_2$, we have $\mathcal{M}_2, w_2 \models \Diamond \bigwedge_{i=1,\ldots,m} \varphi_i$. Therefore, $\mathcal{M}_1, w_1 \models \Diamond \bigwedge_{i=1,\ldots,m} \varphi_i$. But this is a contradiction, since the only states accessible from w_1 are $w^1, \ldots, w^m$, and for each w^i there is a φ_i such that $\mathcal{M}_1, w^i \not\models \varphi_i$. The proof of the zag condition is similar. QED

The modal invariance Lemma (Lemma 3.7) can be used to prove what can and cannot be expressed in the basic modal language.

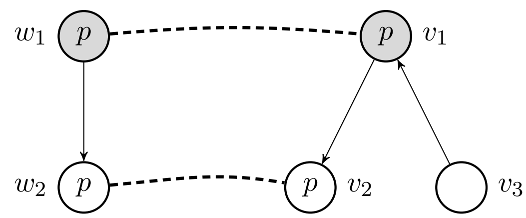
Fact 3.9 *Let $\mathcal{M} = \langle W, R, V\rangle$ be a relational structure. The* universal operator *is a unary operator $\mathsf{A}\varphi$ defined as follows:*

$$\mathcal{M}, w \models \mathsf{A}\varphi \textit{ iff for all } v \in W,\ \mathcal{M}, v \models \varphi$$

The universal operator A is not definable in the basic modal language.

Proof. Suppose that the universal operator is definable in the basic modal language. Then there is a basic modal formula $\alpha(\cdot)$ such[2] that for any formula φ and any relational structure $\mathcal{M}$ with state w, we have $\mathcal{M}, w \models \mathsf{A}\varphi$ iff $\mathcal{M}, w \models \alpha(\varphi)$. Consider the relational structure $\mathcal{M} = \langle W, R, V\rangle$ with $W = \{w_1, w_2\}$, $R = \{(w_1, w_2)\}$ and $V(w_1, p) = V(w_2, p) = 1$. Note that $\mathcal{M}, w_1 \models \mathsf{A}p$. Since the universal operator is assumed to be defined by $\alpha(\cdot)$, we must have $\mathcal{M}, w_1 \models \alpha(p)$. Consider the relational structure $\mathcal{M}' = \langle W', R', V'\rangle$ with $W' = \{v_1, v_2, v_3\}$, $R' = \{(v_1, v_2), (v_3, v_1)\}$ and $V'(v_1, p) = V'(v_2, p) = 1$. Note that $Z = \{(w_1, v_2), (w_2, v_2)\}$ is a bismulation relating w_1 and v_1 (i.e., $\mathcal{M}, w_1 \,\underline{\leftrightarrow}\, \mathcal{M}', v_1$). These relational structures and bisimulation is pictured below:

[2] The notation $\alpha(\cdot)$ means that α is a basic modal formula with "free slots" such that $\alpha(\varphi)$ is a well formed modal formula with φ plugged into the free slots.



By Lemma 3.7, $\mathcal{M}, w_1 \leftrightsquigarrow \mathcal{M}', v_1$. Therefore, since $\alpha(p)$ is a formula of the basic modal language and $\mathcal{M}, w_1 \models \alpha(p)$, we have $\mathcal{M}', v_1 \models \alpha(p)$. Since $\alpha(p)$ defines the universal operator, $\mathcal{M}', v_1 \models \mathsf{A}p$, which is a contradiction. Hence, A is not definable in the basic modal language. QED

Fact 3.10 *Let $\mathcal{M} = \langle W, R, V \rangle$ be a relational structure. Define the "exists two" operator $\Diamond_2\varphi$ as follows:*

$$\mathcal{M}, w \models \Diamond_2\varphi \textit{ iff there is } v_1, v_2 \in W \textit{ such that } v_1 \neq v_2,\ \mathcal{M}, v_1 \models \varphi \textit{ and } \mathcal{M}, v_2 \models \varphi$$

The exist two $\Diamond_2$ operator is not definable in the basic modal language.

Proof. Suppose that the $\Diamond_2$ is definable in the basic modal language. Then there is a basic modal formula $\alpha(\cdot)$ such that for any formula φ and any relational structure $\mathcal{M}$ with state w, we have $\mathcal{M}, w \models \Diamond_2\varphi$ iff $\mathcal{M}, w \models \alpha(\varphi)$. Consider the relational structure $\mathcal{M} = \langle W, R, V \rangle$ with $W = \{w_1, w_2, w_3\}$, $R = \{(w_1, w_2), (w_1, w_3)\}$ and $V(w_2, p) = V(w_3, p) = 1$. Note that $\mathcal{M}, w_1 \models \Diamond_2 p$. Since $\Diamond_2$ is assumed to be defined by $\alpha(\cdot)$, we must have $\mathcal{M}, w_1 \models \alpha(p)$. Consider the relational structure $\mathcal{M}' = \langle W', R', V' \rangle$ with $W' = \{v_1, v_2\}$, $R' = \{(v_1, v_2)\}$ and $V'(v_2, p) = 1$. Note that $Z = \{(w_1, v_2)\}$ is a bismulation relating w_1 and v_1 (i.e., $\mathcal{M}, w_1 \underline{\leftrightarrow} \mathcal{M}', v_1$). By Lemma 3.7, $\mathcal{M}, w_1 \leftrightsquigarrow \mathcal{M}', v_1$. Therefore, since $\alpha(p)$ is a formula of the basic modal language and $\mathcal{M}, w_1 \models \alpha(p)$, we have $\mathcal{M}', v_1 \models \alpha(p)$. Since $\alpha(\cdot)$ defines $\Diamond_2$, $\mathcal{M}', v_1 \models \Diamond_2 p$, which is a contradiction. Hence, $\Diamond_2$ is not definable in the basic modal language. QED

3.1 Defining Classes of Structures

The basic modal language can also be used to define *classes* of structures.

Definition 3.11 (Frame) A pair $\langle W, R \rangle$ with W a nonempty set of states and $R \subseteq W \times W$ is called a **frame**. Given a frame $\mathcal{F} = \langle W, R \rangle$, we say the model $\mathcal{M}$ **is based on the frame** $\mathcal{F} = \langle W, R \rangle$ if $\mathcal{M} = \langle W, R, V \rangle$ for some valuation function V. ◁

Definition 3.12 (Frame Validity) Given a frame $\mathcal{F} = \langle W, R \rangle$, a modal formula φ is **valid on** $\mathcal{F}$, denoted $\mathcal{F} \models \varphi$, provided $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ based on $\mathcal{F}$. ◁

Suppose that P is a property of relations (eg., reflexivity or transitivity). We say a frame $\mathcal{F} = \langle W, R \rangle$ has property P provided R has property P. For example,

- $\mathcal{F} = \langle W, R \rangle$ is called a **reflexive frame** provided R is reflexive, i.e., for all $w \in W$, wRw.
- $\mathcal{F} = \langle W, R \rangle$ is called a **transitive frame** provided R is transitive, i.e., for all $w, x, v \in W$, if wRx and xRv then wRv.

Definition 3.13 (Defining a Class of Frames) A modal formula φ **defines the class of frames with property** P provided for all frames $\mathcal{F}$, $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ has property P. $\triangleleft$

Remark 3.14 *Note that if $\mathcal{F} \models \varphi$ where φ is some modal formula, then $\mathcal{F} \models \varphi^*$ where φ^* is any* **substitution instance** *of φ. That is, φ^* is obtained by replacing sentence letters in φ with modal formulas. In particular, this means, for example, that in order to show that $\mathcal{F} \not\models \Box\varphi \rightarrow$ it is enough to show that $\mathcal{F} \not\models \Box p \rightarrow p$ where p is a sentence letter. (This will be used in the proofs below).*

Fact 3.15 *$\Box\varphi \rightarrow \varphi$ defines the class of reflexive frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Box\varphi \rightarrow \varphi$ iff $\mathcal{F}$ is reflexive.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ is reflexive and let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Box\varphi \rightarrow \varphi$. Suppose that $\mathcal{M}, w \models \Box\varphi$. Then for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$. Since R is reflexive, we have wRw. Hence, $\mathcal{M}, w \models \varphi$. Therefore, $\mathcal{M}, w \models \Box\varphi \rightarrow \varphi$, as desired.

($\Rightarrow$) We argue by contraposition. Suppose that $\mathcal{F}$ is not reflexive. We must show $\mathcal{F} \not\models \Box\varphi \rightarrow \varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Box p \rightarrow p$ for some sentence letter p. Since $\mathcal{F}$ is not reflexive, there is a state $w \in W$ such that it is not the case that wRw. Consider the model $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$ with $V(v, p) = 1$ for all $v \in W$ such that $v \neq w$. Then $\mathcal{M}, w \models \Box p$ since, by assumption, for all $v \in W$ if wRv, then $v \neq w$ and so $V(v, p) = 1$. Also, notice that by the definition of V, $\mathcal{M}, w \not\models p$. Therefore, $\mathcal{M}, w \models \Box p \wedge \neg p$, and so, $\mathcal{F} \not\models \Box p \rightarrow p$. QED

Fact 3.16 *$\Box\varphi \rightarrow \Box\Box\varphi$ defines the class of transitive frames.*

Proof. We must show for any frame $\mathcal{F}$, $\mathcal{F} \models \Box\varphi \rightarrow \Box\Box\varphi$ iff $\mathcal{F}$ is transitive.

($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ is transitive and let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$. Given $w \in W$, we must show $\mathcal{M}, w \models \Box\varphi \rightarrow \Box\Box\varphi$. Suppose that $\mathcal{M}, w \models \Box\varphi$. We must show $\mathcal{M}, w \models \Box\Box\varphi$. Suppose that $v \in W$ and wRv. We must show $\mathcal{M}, v \models \Box\varphi$. To that end, let $x \in W$ be any state with vRx. Since R is transitive and wRv and vRx, we have wRx. Since $\mathcal{M}, w \models \Box\varphi$, we have $\mathcal{M}, x \models \varphi$. Therefore, since x is an arbitrary state accessible from v, $\mathcal{M}, v \models \Box\varphi$. Hence, $\mathcal{M}, w \models \Box\Box\varphi$, and so, $\mathcal{M}, w \models \Box\varphi \rightarrow \Box\Box\varphi$, as desired.

($\Rightarrow$) We argue by contraposition. Suppose that $\mathcal{F}$ is not transitive. We must show $\mathcal{F} \not\models \Box\varphi \rightarrow \Box\Box\varphi$. By the above Remark, it is enough to show $\mathcal{F} \not\models \Box p \rightarrow \Box\Box p$ for some sentence letter p. Since $\mathcal{F}$ is not transitive, there are states $w, v, x \in W$ with wRv and vRx but it is not the case that wRx. Consider the model $\mathcal{M} = \langle W, R, V\rangle$ based on $\mathcal{F}$ with $V(y, p) = 1$ for all $y \in W$ such that $y \neq x$. Since $\mathcal{M}, x \not\models p$ and wRv and vRx, we have $\mathcal{M}, w \not\models \Box\Box p$. Furthermore, $\mathcal{M}, w \models \Box p$ since the only state where p is false is x and it is assumed that it is not the case that wRx. Therefore, $\mathcal{M}, w \models \Box p \wedge \neg\Box\Box p$, and so, $\mathcal{F} \not\models \Box p \rightarrow \Box\Box p$, as desired. QED

Question 3.17 *Determine which class of frames are defined by the following modal formulas (prove your answer).*

1. $\Box\varphi \rightarrow \Diamond\varphi$

2. $\Diamond\varphi \to \Box\varphi$

3. $\varphi \to \Box\Diamond\varphi$

4. $\Box(\Box\varphi \to \varphi)$

5. $\Diamond\Box\varphi \to \Box\Diamond\varphi$

4 The Minimal Modal Logic

For a complete discussion of this material, consult Chapter 5 of *Modal Logic for Open Minds* by Johan van Benthem.

Definition 4.1 (Substitution) A **substitution** is a function from sentence letters to well formed modal formulas (i.e., $\sigma : \mathcal{S} \to \mathcal{L}$). We extend a substitution σ to all formulas φ by recursion as follows (we write φ^σ for $\sigma(\varphi)$):

1. $\sigma(\bot) = \bot$
2. $\sigma(\neg\varphi) = \neg\sigma(\varphi)$
3. $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$
4. $\sigma(\Box\varphi) = \Box\sigma(\varphi)$
5. $\sigma(\Diamond\varphi) = \Diamond\sigma(\varphi)$ $\triangleleft$

For example, if $\sigma(p) = \Box\Diamond(p \wedge q)$ and $\sigma(q) = p \wedge \Box q$ then

$$(\Box(p \wedge q) \to \Box p)^\sigma = \Box((\Box\Diamond(p \wedge q)) \wedge (p \wedge \Box q)) \to \Box(\Box\Diamond(p \wedge q))$$

Definition 4.2 (Tautology) A modal formula φ is called a **(propositional) tautology** if $\varphi = (\alpha)^\sigma$ where σ is a substition, α is a formula of propositional logic and α is a tautology. $\triangleleft$

For example, $\Box p \to (\Diamond(p \wedge q) \to \Box p)$ is a tautology because $a \to (b \to a)$ is a tautology in the language of propositional logic and

$$(a \to (b \to a))^\sigma = \Box p \to (\Diamond(p \wedge q) \to \Box p)$$

where $\sigma(a) = \Box p$ and $\sigma(b) = \Diamond(p \wedge q)$.

Definition 4.3 (Modal Deduction) A **modal deduction** is a finite sequence of formulas $\langle\alpha_1, \ldots, \alpha_n\rangle$ where for each $i \leq n$ either

1. α_i is a tautology
2. α_i is a substitution instance of $\Box(p \to q) \to (\Box p \to \Box q)$
3. α_i is of the form $\Box\alpha_j$ for some $j < i$
4. α_i follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$).

We write $\vdash_{\mathbf{K}} \varphi$ if there is a deduction containing φ. $\triangleleft$

The formula in item 2. above is called the **K axiom** and the application of item 3. is called the rule of **necessitation**.

Fact 4.4 $\vdash_{\mathbf{K}} \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$

Proof.

1.	$\varphi \wedge \psi \to \varphi$	tautology
2.	$\Box((\varphi \wedge \psi) \to \varphi)$	Necessitation 1
3.	$\Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\varphi)$	Substitution instance of K
4.	$\Box(\varphi \wedge \psi) \to \Box\varphi$	MP 2,3
5.	$\varphi \wedge \psi \to \psi$	tautology
6.	$\Box((\varphi \wedge \psi) \to \psi)$	Necessitation 5
7.	$\Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\psi)$	Substitution instance of K
8.	$\Box(\varphi \wedge \psi) \to \Box\psi$	MP 5,6
9.	$(a \to b) \to ((a \to c) \to (a \to (b \wedge c)))$	tautology $(a := \Box(\varphi \wedge \psi), b := \Box\varphi, c := \Box\psi)$
10.	$(a \to c) \to (a \to (b \wedge c))$	MP 4,9
11.	$\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$	MP 8,10

QED

Fact 4.5 *If* $\vdash_{\mathbf{K}} \varphi \to \psi$ *then* $\vdash_{\mathbf{K}} \Box\varphi \to \Box\psi$

Proof.

1.	$\varphi \to \psi$	assumption
2.	$\Box(\varphi \to \psi)$	Necessitation 1
3.	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	Substitution instance of K
4.	$\Box\varphi \to \Box\psi$	MP 2,3

QED

Definition 4.6 (Modal Deduction with Assumptions) Let Σ be a set of modal formulas. A **modal deduction of** φ **from** Σ, denoted $\Sigma \vdash_{\mathbf{K}} \varphi$ is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. α_i is a tautology
2. $\alpha_i \in \Sigma$
3. α_i is a substitution instance of $\Box(p \to q) \to (\Box p \to \Box q)$
4. α_i is of the form $\Box\alpha_j$ for some $j < i$ and $\vdash_{\mathbf{K}} \alpha_j$
5. α_i follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that α_k is of the form $\alpha_j \to \alpha_i$). $\triangleleft$

Remark 4.7 *Note that the side condition in item 4. in the above definition is crucial. Without it, one application of Necessitation shows that* $\{p\} \vdash_{\mathbf{K}} \Box p$*. Using the Deduction Theorem, we have* $\Sigma \cup \{\alpha\} \vdash_{\mathbf{K}} \beta$ *implies* $\Sigma \vdash_{\mathbf{K}} \alpha \to \beta$*, we can conclude that* $\vdash_{\mathbf{K}} p \to \Box p$*. But, clearly* $p \to \Box p$ *cannot be a theorem (why?).*

Definition 4.8 (Logical Consequence) Suppose that Σ is a set of modal formulas. We say φ is a **logical consequence** of Σ, denoted $\Sigma \models \varphi$ provided for all frames $\mathcal{F}$, if $\mathcal{F} \models \alpha$ for each $\alpha \in \Sigma$, then $\mathcal{F} \models \varphi$. $\triangleleft$

Theorem 4.9 (Soundness) *If* $\Sigma \vdash_{\mathbf{K}} \varphi$ *then* $\Sigma \models \varphi$.

Proof. The proof is by induction on the length of derivations. See Chapter 5 in *Modal Logic for Open Minds* and your lecture notes. QED

Theorem 4.10 (Completeness) *If* $\Sigma \models \varphi$ *then* $\Sigma \vdash_{\mathbf{K}} \varphi$.

Proof. See Chapter 5 in *Modal Logic for Open Minds* and your lecture notes for a proof. QED

Remark 4.11 (Alternative Statement of Soundness and Completeness) *Suppose that* Σ *is a set of modal formulas. Define the minimal modal logic as the smallest set* $\Lambda_{\mathbf{K}}(\Sigma)$ *of modal formulas extending* Σ *that (1) contains all tautologies, (2) contains the formula* $\Box(p \to q) \to (\Box p \to \Box q)$*, (3) is closed under substitutions, (4) is closed under the Necessitation rule (i.e., if* $\varphi \in \Lambda_{\mathbf{K}}$ *is derivable* without premises – $\vdash_{\mathbf{K}} \varphi$ – *then* $\Box\varphi \in \Lambda_{\mathbf{K}}$*) and (4) is closed under Modus Ponens. Suppose* $\mathfrak{F}(\Sigma) = \{\varphi \mid \Sigma \models \varphi\}$*. Then, soundness and completeness states that* $\Lambda_{\mathbf{K}}(\Sigma) = \mathfrak{F}(\Sigma)$*.*

5 Proof of Completeness

We start by reminding you of the key definitions and facts about *maximally consistent sets* (recall the Fitting notes on propositional logic).

- Let $\mathbf{K}$ denote the minimal modal logic and $\vdash \varphi$ mean φ is derivable in $\mathbf{K}$. If Γ is a set of formulas, we write $\Gamma \vdash \varphi$ if $\vdash (\psi_1 \wedge \cdots \wedge \psi_k) \to \varphi$ for some finite set $\psi_1, \ldots, \psi_k \in \Gamma$.
- Let Γ be a set of formulas. If $\mathcal{F}$ is a frame, then we write $\mathcal{F} \models \Gamma$ for $\mathcal{F} \models \varphi$ for each $\varphi \in \Gamma$. We write $\Gamma \models \varphi$ provided for all frames $\mathcal{F}$, if $\mathcal{F} \models \Gamma$ then $\mathcal{F} \models \varphi$.
- A set of formulas Γ is **consistent** provided $\Gamma \not\vdash \bot$.
- Γ is a **maximally consistent set** if Γ is consistent and for each $\varphi \in \mathcal{L}$ either $\varphi \in \Gamma$ of $\neg\varphi \in \Gamma$. Alternatively, Γ is consistent and every Γ' such that $\Gamma \subseteq \Gamma'$ is inconsistent.
- A logic is strongly complete if $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$. It is weakly complete if $\models \varphi$ implies $\vdash \varphi$. Strong completeness implies weak completeness, but weak completeness does not imply strong completeness.

Recall the following key facts about maximally consistent sets. Suppose that Γ is a maximally consistent set,

1. If $\vdash \varphi$ then $\varphi \in \Gamma$
2. If $\varphi \to \psi \in \Gamma$ and $\varphi \in \Gamma$ then $\psi \in \Gamma$
3. $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$

4. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$
5. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$

Lemma 5.1 (Lindenbaum's Lemma) *For each consistent set Γ, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. In other words, every consistent set Γ can be extended to a maximally consistent set.*

Definition 5.2 (Canonical Model) The canonical model for **K** is the model $\mathcal{M}^c = \langle W^c, R^c, V^c \rangle$ where

- $W^c = \{\Gamma \mid \Gamma \text{ is a maximally consistent set}\}$
- $\Gamma R^c \Delta$ iff $\Gamma^\Box = \{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta$
- $V^c(p) = \{\Gamma \mid p \in \Gamma\}$ $\triangleleft$

Lemma 5.3 (Truth Lemma) *For every $\varphi \in \mathcal{L}$, $\mathcal{M}^c, \Gamma \models \varphi$ iff $\varphi \in \Gamma$*

Proof. The proof is by induction on the structure of φ.
Base Case: φ is an atomic proposition p. Then,

$$\begin{array}{llll} \mathcal{M}^c, \Gamma \models p & \text{iff} & \Gamma \in V^c(p) & \text{(Def. of Truth)} \\ & \text{iff} & p \in \Gamma & \text{(Def. of } V^c\text{)} \end{array}$$

Induction Step: There are three cases:
Case 1: φ is $\psi_1 \wedge \psi_2$.

$$\begin{array}{llll} \mathcal{M}^c, \Gamma \models \psi_1 \wedge \psi_2 & \text{iff} & \mathcal{M}^c, \Gamma \models \psi_1 \text{ and } \mathcal{M}^c, \Gamma \models \psi_2 & \text{(Def. of Truth)} \\ & \text{iff} & \psi_1 \in \Gamma \text{ and } \psi_2 \in \Gamma & \text{(Induction Hypothesis)} \\ & \text{iff} & \psi_1 \wedge \psi_2 \in \Gamma & \text{(Property 4. of Max Consistent Sets)} \end{array}$$

Case 2: φ is $\neg\psi$

$$\begin{array}{llll} \mathcal{M}^c, \Gamma \models \neg\psi & \text{iff} & \mathcal{M}^c, \Gamma \not\models \psi & \\ & \text{iff} & \psi \notin \Gamma & \text{(Induction Hypothesis)} \\ & \text{iff} & \neg\psi \in \Gamma & \text{(Property 3. of Max Consistent Sets)} \end{array}$$

Case 3: φ is $\Box\psi$. Suppose that $\Box\psi \in \Gamma$. Then for each Δ, if $\Gamma R^c \Delta$, then $\psi \in \Gamma^\Box \subseteq \Delta$. By the induction hypothesis, $\mathcal{M}^c, \Delta \models \psi$. Since this is true for any Δ with $\Gamma R^c \Delta$, we have $\mathcal{M}^c, \Gamma \models \Box\psi$.

Suppose that $\Box\psi \notin \Gamma$. We claim that $\Gamma^\Box \cup \{\neg\psi\}$ is consistent. Suppose not. Then $\Gamma^\Box \cup \{\neg\psi\} \vdash \bot$. This means, $\vdash (\alpha_1 \wedge \cdots \wedge \alpha_n \wedge \neg\psi) \to \bot$ with each $\alpha_i \in \Gamma^\Box$. By propositional reasoning[3], we have $\vdash (\alpha_1 \wedge \cdots \wedge \alpha_n) \to \psi$. Using Facts 4.4 and 4.5, we have

$$\vdash (\Box\alpha_1 \wedge \cdots \wedge \Box\alpha_n) \to \Box\psi$$

Since each $\Box\alpha_i \in \Gamma$, we have $\Box\alpha_1 \wedge \cdots \wedge \Box\alpha_n \in \Gamma$, and so $\Box\psi \in \Gamma$. This contradicts our assumption. Hence, $\Gamma^\Box \cup \{\neg\psi\}$ is consistent. By Lindenbaum's Lemma, there is a maximally consistent set Δ with $\Gamma^\Box \cup \{\neg\psi\} \subseteq \Delta$. By the induction hypothesis, $\mathcal{M}^c, \Delta \not\models \psi$. Furthermore, we have $\Gamma R^c \Delta$. Hence, $\mathcal{M}^c, \Gamma \not\models \Box\psi$, as desired. QED

[3] Here we use the fact that, in propositional logic, if $\vdash (\varphi \wedge \neg\psi) \to \bot$, then $\vdash \varphi \to \psi$.

Theorem 5.4 *Every maximally consistent set* Γ *has a model (i.e., there is a models* $\mathcal{M}$ *and state* w *such that for all* $\varphi \in \Gamma$*,* $\mathcal{M}, w \models \varphi$*.*

Proof. Suppose that Γ is a consistent set. By Lindenbaum's Lemma, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. Then, by the Truth Lemma, for each $\varphi \in \Gamma'$, we have $\mathcal{M}^c, \Gamma' \models \varphi$. Then, in particular, every formula in Γ is true at Γ' in the canonical model. QED

Theorem 5.5 *If* $\Gamma \models \varphi$ *then* $\Gamma \vdash \varphi$

Proof. Suppose that $\Gamma \not\vdash \varphi$. Then, $\Gamma \cup \{\neg\varphi\}$ is consistent. By the above theorem, there is a model of $\Gamma \cup \{\neg\varphi\}$. Hence, $\Gamma \not\models \varphi$. QED